On Monopoly Insurance Pricing when Agents Differ in Risk Aversion

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Abstract: The paper analyzes a monopolistic insurer’s pricing strategies when potential customers differ in risk aversion and their type cannot be observed by the insurer. Our model builds on Schlesinger (1983), who derived optimal nonlinear pricing strategies for competitive and monopolistic insurance markets. While Schlesinger assumed existence, we are concerned with conditions under which optimal strategies may exist. We introduce a general model framework for continuous but not necessarily differentiable utility functions and derive conditions for existence of optimal insurance pricing strategies. An important application of our findings is “kinked” utility functions, which are found to offer a better match of actual decision making. Both fixed and proportionate premium loadings (relative to expected loss) are considered. [Key words: insurance monopoly; “kinked” utility; nonlinear pricing]

INTRODUCTION

Traditional models of insurance demand refer to the relationship of a representative insurer with a representative customer. The maximum premium the customer is prepared to pay for (full) insurance coverage is known to the insurer. Under this assumption, a monopolistic insurer maximizes its expected profit and offers full insurance to the customer who

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pays his reservation price in exchange. This reservation price is the expected loss of the policy plus the individual risk premium.

We extend the simple model framework of only two market participants (i.e., an insurer and a representative customer) to a market with many different potential customers looking for insurance. In the standard model, it is implicitly assumed that the insurer has perfect information about individual risk preferences (i.e. utility functions) of potential customers so that individually optimal prices can be set. This assumption seems very restrictive. While asymmetric information regarding consumer risk-aversion has no effect in a competitive insurance market, risk-aversion matters for a monopolist who seeks to extract maximum consumer surplus (Stiglitz, 1977: 428).

Empirical findings suggest that individuals tend to make different decisions in identical risk situations—i.e., they value risk situations differently, even if they are endowed with similar initial wealth. In other words, in a given population of potential customers, we may generally find different risk preferences. Individuals with higher risk-aversion tend to have a higher willingness to pay for full insurance than individuals with less risk-aversion. In this article, we let consumers differ in risk preferences but consider an equal level of exogenously determined risk they are exposed to. The insurer cannot observe individual risk preferences, but it does know the distribution of characteristics among the population of potential customers, which might be so, for instance, by experience. This more realistic case has not been studied in detail in the insurance literature yet. The present work makes an attempt to fill this gap. Our model builds on Schlesinger (1983), who derived optimal nonlinear pricing strategies for competitive and monopolistic insurance markets. While Schlesinger assumed existence of the profit maximizing strategy and differentiability of the insurance coverage function, we are concerned with conditions ensuring that optimal strategies may exist. We derive optimality conditions under which the insurance coverage function need only be continuous (but not differentiable). Given that the set of continuous functions is a superset of differentiable functions, this generalization allows us to include “kinked” utility functions. These utility functions are found to offer a better match of actual decision making. Our model framework applies to first-order risk-aversion occurring at isolated wealth levels (Segal and Spivak, 1990: 118). Allowing for kinked utility is possible only in such a model framework since kinked utility leads to a non-differentiable function of insurance coverage.

Generally, an increase in the price for insurance involves a decrease in demand. Since each potential insured has a maximum individual (risk) premium he is willing to pay for (full) insurance, in our framework the
number of policyholders depends upon the insurance premium offered by the insurer. In view of this functional relationship between price and resulting number of policyholders, we introduce a continuous model. The “optimal” pricing strategy of the insurer consists in identifying the profit-maximizing premium. This strategy must take into account the trade-off between the number of purchasers and expected profit per insurance policy. The intention of this article is to show the existence of optimal pricing strategies—in case of fixed, proportionate, and a combination of both well-known premium loadings—under general assumptions.

The article is organized as follows. In the next section, we briefly comment on the literature background of our analysis. The following section presents the model framework. In a model with incomplete information—i.e., in a setting where the probability distribution of risk preferences is known to the insurer, but not individual risk preferences—we derive the profit function of the insurer and determine the optimal pricing rule. We begin by the simplest case of a fixed premium loading imposed by the insurer. We then introduce proportionate loadings and partial insurance. Finally, we analyze the general insurance demand function given that a combination of both premium loadings (i.e., a fixed loading fee and a proportionate loading factor on the expected value of the policy) is imposed by the insurer.

RELATED LITERATURE

Since the early empirical research by Greer (1974), it is accepted that existing utility theory may sometimes fail to model adequately the decision process of individuals. Greer and Skekel (1975), following Hoskins (1975), argue that actual decision-making might be resulting from a utility function of nonclassical shape—one with a “kink”—in contrast to the classical utility function attributed to von Neumann and Morgenstern (Greer and Skekel, 1975: 843). A reason might be that individuals tend to be more risk-averse (i.e., the utility function might be steeper) up to a certain reference wealth level. “Kinked utility” is also discussed by Sinn (1982), though in a different context. To our knowledge, this interesting empirical result has not been touched yet by insurance economists in the literature. We introduce a general model framework for continuous but not necessarily differentiable utility functions—those with a “kink”—and show the existence of optimal insurance pricing strategies.

This contribution may be embedded in the existing literature on insurance economics as follows. The model framework is most related to Doherty (1975), Stiglitz (1977), and Schlesinger (1983). Doherty (1975)
presents a basic model of insurance demand, building on earlier classics of Mossin (1968) and Smith (1968). He analyzes the optimal contract under different pricing strategies of the insurer—that is, under a fixed loading fee and a loading proportionate to the actuarial value of the policy. In the case of a fixed loading fee, full insurance or no insurance is optimally chosen. In contrast, in the case of a proportionate loading, partial insurance coverage (depending on the loading and the individual utility function) is optimal. We differ from this traditional approach by analyzing the optimal premium for a given probability distribution of risk preferences in a considered customer pool. This approach seems reasonable, since an insurer will not, in general, be informed about the exact risk premium or exact willingness to pay for insurance of a particular customer.

Stiglitz (1977) analyzes the demand for insurance when types of potential customers differ in risk-aversion using a two-type framework. He finds that—given that both types of customers purchase insurance coverage—either both types purchase the same full insurance policy, or the highly risk-averse type purchases full insurance while the low risk-averse type purchases partial insurance, depending on the proportion of highly risk-averse individuals in the population of customers.

Our analysis builds heavily on Schlesinger (1983), who derives optimality conditions for insurance pricing in monopolistic and competitive insurance markets. While Schlesinger derives optimality conditions, it is not clear whether these optimal pricing rules always exist. We derive sufficient conditions in order to ensure existence of such an optimal pricing strategy. Furthermore, our setting is more general since it allows the resulting function of individually optimal proportion of insurance (coinsurance amount) to be non-differentiable. However, Schlesinger is not concerned with the question which pricing strategy is actually best for the insurer in the sense that it brings about highest expected profit. We deal with this question and compare different pricing strategies.

There are a few other economic models of optimal insurance pricing that are related to the one presented here. One is by Kliger and Levikson (1998), who are concerned with the optimal premium and number of policyholders that minimizes the expected loss due to potential insolvency of the insurer. The authors analyze a fixed premium loading in a discrete setting. Schlesinger (1987) considers the expected monopoly profit when the seller offers state-claims contracts to a risk-averse individual with state-dependent preferences. Landsberger and Meilijson (1994) examine the demand for insurance in a model with two types of risk-averse customers. Cleeton and Zellner (1960) use a comparative-static analysis in order to show how the degree of risk-aversion of a consumer, the specification of
the loss, and the price of insurance interact with income to affect the individual’s net demand for insurance.

As we have illustrated, a population of potential policyholders (individuals and firms) is not, as frequently accepted, homogeneous, but rather heterogeneous, since the individuals or firms generally differ in both expected loss and risk preferences. However, in the first case actuarial classification criteria may be taken into account in order to divide policyholders into relatively homogeneous groups with almost equal expected loss. Then, the insurer may price each group separately (Ramsay, 2005: 38). A more detailed discussion on risk classification is offered by Finger (1996). But in the second case, given that the insurer has limited information about individual risk preferences but knows individual expected loss, it is not easily possible to divide policyholders into homogeneous groups. As a result, we will focus our analysis on a population of potential policyholders with identical expected loss but different risk preferences. In such a setting, the optimization problem of the insurer is to find an optimal premium for this customer pool.

MODEL

Our general model of insurance demand builds upon the following basic assumptions. A risk neutral insurer is faced with a continuum of potential customers. Customers are identical in expected loss, but differ in their risk preferences—i.e. they differ in their risk premium \( r \). Each customer purchases one insurance contract at most. The different risk preferences in the considered customer pool may be interpreted in analogy to general demand theory as the sum of individual willingness to pay for full insurance—more specifically, the net individual willingness to pay that exceeds the expected value of the policy. Those may be set up in ascending order to result in the density function of risk preferences. In this case, the insurer cannot skim all consumer rent but may maximize its profit given the known probability distribution of risk preferences in the considered pool of potential insurance buyers.

The insurer may set a premium following the structure

\[
P = \alpha EX + m,
\]

where \( EX \) denotes expected loss of the policy, \( \alpha \) denotes the proportion of insurance, and \( m \) denotes a fixed loading fee to cover the insurer’s expenses. Common wisdom suggests (see, for instance, the seminal articles by Arrow (1963), Mossin (1968), and Doherty (1975)) that the loading \( m \) represents a “lump sum tax” that applies to all consumers but whose imposition might be avoided if no insurance is purchased. Therefore, the optimal insurance coverage for a consumer is either full insurance, \( \alpha = 1 \),
or no insurance, \( \alpha = 0 \) (for a demonstration, the reader is referred to Doherty (1975: 452)). If the insurer knew the risk premium of each customer, it would achieve a complete discrimination in prices by setting the premium exactly equal to the amount the individual potential customer is prepared to pay—i.e., his individual gross reservation price. This outcome results when the insurer knows the utility function of each potential insured. As mentioned above, this will not probably be the case since a complete discrimination in prices will generally involve high transaction cost. Therefore, it seems realistic to assume that the insurer has incomplete information about individual risk preferences, or alternatively that a complete discrimination in prices may indeed be “too expensive” for the insurer to be reasonably undertaken.

Clearly, the insurer may also set its premium according to the structure 
\[
P = aqEX,\]
where \( q \) indicates a proportionate loading factor. Here, the potential customer has to pay a proportionate loading factor upon the expected value of the policy, so that the individual utility function determines the coinsurance rate he will actually choose. Therefore, the utility function determines both the risk premium and the individually chosen coinsurance rate. The loading factor is again the same for all potential customers.

Finally, the insurer may set a premium following the general structure 
\[
P = aqEX + m,\]
where we allow for fixed loading fee and proportionate loading factor simultaneously. In this case, the premium structure incorporates the potential to extract expected profit from both fixed loading fee and proportionate loading factor. Therefore, this general pricing strategy seems to be a powerful instrument.

In the sequel, we will address each possibility separately. Given that the insurer knows the distribution of risk preferences in the customer pool, we will see that a general insurance demand function can always be derived. Even though the insurer cannot distinguish individual risk preferences, it can nevertheless determine an optimal price. We analyze a situation similar to the classical monopoly model. Interestingly, the model might also be of concern in all situations where an insurer exerts some market power—i.e., in all cases where price-setting behavior on the part of the insurer is important. Note that in Germany, for instance, some insurers have a market share of more than 10\%, which suggests that market power might indeed play an important role in insurance markets (Schradin, 2003: 613).

Suppose probability density of individuals’ risk preferences is given by \( f(r) \) over some closed interval \( r \in [0, \bar{r}] \). Individuals thus differ in their risk preferences \( r \). Hence, at one extreme, we have a maximum risk premium \( \bar{r} > 0 \) in the customer pool, and at the other extreme, we find the least
risk-averse individual to be actually risk neutral, \( r = 0 \). By these assumptions, we include all potential policyholders in the analysis, since risk loving individuals (those with \( r < 0 \)) would not ask for insurance. Throughout our analysis, we assume that the insurer knows the distribution \( f(r) \) across the population of potential policyholders, but it cannot determine the \( r \)-type of any individual prior to insurance purchase. An individual’s \( r \)-type is private information. The risk premium \( r \), as we use it here, was introduced by Pratt (1964). Remember that for any strictly concave utility function \( U(\cdot) \) the risk premium \( r \) is defined via the equation
\[
pU(W - L) + (1 - p)U(W) = U(W - pL - r) .
\]
The risk premium \( r \) denotes the maximum an agent is willing to pay to securely receive the expected value of the probability distribution instead of the probability distribution itself.

We make the same basic assumptions as in the basic model of insurance demand. All potential policyholders dispose of initial wealth \( W \) and face a potential loss \( L < W \) that occurs with probability \( p \). Thus, the size of loss is a random variable with expected value \( pL \) (this assumption is not restrictive since our analysis also holds in the non-binary case, however, for notational convenience, we examine the binary case here). In order to ensure non-negative expected profits of the insurer, the insurance premium must not be lower than the expected value of the policy. Furthermore, the premium should not exceed the reservation price of the most risk-averse customer in the customer pool (i.e., the maximum risk premium \( \hat{r} \)), since then there would be no demand for insurance. As a consequence, the profit-maximizing premium of the insurer, \( P^* \), will lie in the interval \([pL, pL + \hat{r}]\).

**FIXED LOADINGS**

In this section, we begin our analysis with the case where individuals choose either full insurance or no insurance at all. That is, we consider a fixed premium loading \( m \).

For expositional convenience, we normalize the number of potential policyholders to one. Then, let \( x(m) \in [0,1] \) be the fraction of customers purchasing insurance given that the “net insurance premium” is \( m \). This so-called economic premium presents the net price of insurance, adjusted to the expected value of the policy, as noted in Kliger and Levikson (1998: 245). Hence, the gross insurance premium \( P \) is determined by
\[
m = P - pL .
\]
Insurance demand is given by
\[
x(m) = \int_m^{\hat{r}} f(r)dr = 1 - F(m) ,
\]
where \( F(\cdot) \) is the probability distribution according to the density \( f(\cdot) \). When \( m \) tends to zero, all risk-averse customers will purchase insurance. In order to further analyze the function \( x(m) \) in (1), we consider the points of axis interception:

\[
x(0) = 1 - F(0) = 1
\]

\[
0 = x(m) = 1 - F(m) \iff F(m) = 1 \iff m = r.
\]

All potential policyholders purchase insurance at a fair premium \( m = 0 \), while nobody would ask for insurance at a net price \( m = r \). Furthermore, the function is injective (one-to-one), since we have \( x'(m) = -f(m) < 0 \) with \( m \in [0,r] \). Therefore, the inverse demand function always exists and is given by

\[
m(x) = F^{-1}(1-x) \text{ with } x \in [0,1].
\]

This function is strictly decreasing due to

\[
m'(x) = \frac{1}{f(F^{-1}(1-x))} < 0.
\]

We may calculate the expected profit of the insurer as

\[
\Pi(m) = (m - c) \cdot x(m) \text{ with } m \in [0,r],
\]

where \( c < r \) is marginal cost. The necessary condition of an interior maximum of the profit function is given by

\[
\Pi'(m) = x(m) + (m - c) \cdot x'(m) = [1 - F(m)] - (m - c) \cdot f(m) = 0.
\]

\( 1 - F(m) \) represents the number of customers who purchase insurance coverage at net premium \( m \). Increasing \( m \) by one dollar would bring in \( 1 - F(m) \) more revenue if demand did not change. Yet, \( f(m) \) represents the number of customers who would “drop out” of the market if the loading increased by one dollar (i.e., \( x'(m) = -f(m) \) from (1)). The monopolist would lose \( m \) for each one of these customers in revenue, but it also would not need to bear the cost of providing insurance \( c \). The marginal profit for increasing \( m \) in (5) should, of course, be zero at the optimum.
As a result, normalizing marginal cost to zero, the profit maximizing net premium of the insurer is implicitly defined by

\[ m^* = \frac{1 - F(m^*)}{f(m^*)}, \]  

(6)

which implies the optimal (i.e., the profit maximizing) gross premium of the insurer

\[ P^* = pL + m^*. \]  

(7)

In the following, we discuss the sufficient condition for a maximum. We distinguish three cases. First, in case \( \Pi(\cdot) \) is twice differentiable at \( m^* \), a sufficient condition for a maximum in \( m^* \) is

\[ f'(m^*) > -\frac{2}{m^*} \cdot f(m^*), \]

(8)

which means that \( f(\cdot) \) should not be heavily decreasing in \( m^* \). From an economic viewpoint, this seems to be fulfilled given that most people tend to be rather risk-averse while risk attitudes approaching risk neutrality are relatively uncommon. Therefore, we may argue that density of risk premia is broadly increasing. Second, in case \( \Pi(\cdot) \) is only once differentiable at \( m^* \), a sufficient condition is that \( \Pi(\cdot) \) has a change of sign from positive to negative and is thus maximal at \( m^* \). To see this, observe that if \( m < m^* \)

\[ \Pi'(m) = \Pi(m) - \Pi(m^*) = F(m^*) - F(m) + m^*f(m^*) - mf(m) \]

(9)

is positive as long as \( f(\cdot) \) is increasing, i.e., \( f(\cdot) \) not heavily decreasing, in \( m^* \). If, in contrast, \( m > m^* \) then (9) is negative in the same case. Finally, in case \( \Pi(\cdot) \) is not differentiable at \( m^* \), for a maximum at \( m^* \), a sufficient condition is that

\[ f(m^* + h) = \begin{cases} f(m^*) + h \cdot M + o(h) & \text{if } h \to 0, h \geq 0, M < 0, \\ f(m^*) + h \cdot M + o(h) & \text{if } h \to 0, h \leq 0, M > 0, \end{cases} \]

(10)

which implies the existence of the limit of the difference quotient from the left and from the right with respective sign (the symbol \( o(h) \) indicates the Landau symbol, meaning that \( o(h) \) approaches zero faster than \( h \)).
Since \( x(m) \) is a continuous function due to (1), the expected profit of the insurer is a continuous function in \( m \). We maximize over the compact interval \([0, r]\), so that an optimal premium maximizing (4) always exists—this is because continuous functions over a compact interval always have a maximum as shown in the "extreme value theorem" (Dugundji, 1970: 227).

In summary, we have shown that—for any given continuous probability distribution of risk preferences of potential insurance purchasers—an optimal premium for the insurer always exists. This optimal premium does not necessarily need to be unique. When the insurer is informed about the distribution of risk preferences in the customer pool, but cannot distinguish individual customers, it can nevertheless always determine an optimal price by taking into account the resulting insurance demand function.

**PROPORTIONAL LOADINGS**

In this section, we focus our attention on a setting where the insurer determines its price according to a premium structure that includes a loading factor proportional to the expected value of the policy. Within this framework, we do not restrict our analysis to full insurance any more, but include partial insurance coverage. If the insurer offers a proportionate premium loading, the expected-utility maximizing customers will not necessarily want to fully insure (or even not insure at all), but instead will choose partial insurance coverage in optimum.

If the insurer calculates its premium on the basis of a proportionate loading factor \( q \geq 1 \) upon the actuarial value of the policy, the premium structure becomes \( P = \alpha qpL \), where \( 0 \leq \alpha \leq 1 \) indicates the coinsurance rate. A policyholder chooses individual coinsurance \( \alpha \) which maximizes his expected utility of final wealth, i.e.,

\[
\max EU(W) = (1 - p) \cdot U(W - \alpha qpL) + p \cdot U(W - \alpha qpL - (1 - \alpha)L),
\]  

(11)

where \( W \) denotes initial wealth. If the utility function is differentiable, the well-known first order condition for an interior maximum is given by

\[
\frac{1 - p U'(W_1)}{p} = \frac{1 - q p}{qp},
\]

(12)

where \( W_1 \) and \( W_2 \) represent final wealth of the insured in the no-loss state and the loss state, respectively.
Since the risk premium $r$ depends upon an individual’s utility function and also the lottery he faces, which is here in fact actually chosen by him, the risk premium is not an exogenous parameter. Therefore, we may formulate an individual’s utility function as $U_\theta$, where $\theta \in [\underline{\theta}, \bar{\theta}]$ is indeed an exogenous parameter indicating the curvature of the utility function. To simplify notation, we write $U(\theta)$ as $U_\theta$. Then the demand for insurance or, more precisely, individually optimal coinsurance is given by $\alpha^*(U_\theta,q)$, which results implicitly from (12) above. For any insurer-determined proportionate loading factor $q$, an individual with utility $U_\theta$ will choose an individually optimal coinsurance rate $\alpha^*(U_\theta,q)$, i.e.,

$$\alpha^*(U_\theta,q) = \arg\max_{\alpha} EU_\theta(W).$$  \hspace{1cm} (13)

The sales of the insurer are determined by the integral over the density-weighted aggregate insurance demand

$$x(q) = \int_{\underline{\theta}}^{\bar{\theta}} \alpha^*(U_\theta,q)f(\theta)d\theta. \hspace{1cm} (14)$$

Hence, we have shown that the sales of the insurer and thus the general insurance demand function may be depicted as a classical expected value of random variable $\alpha^*(U_\theta,q)$. Note that when the proportionate premium loading of the insurer $q$ approaches one (i.e., the premium becomes actuarially fair), the individually optimal coinsurance rate $\alpha^*(U_\theta,q)$ also approaches one. Then, all potential policyholders indeed ask for insurance and choose full insurance in optimum. In order to ensure the existence of the integral in (14) and in the following, we need two general conditions for the function $\alpha_\theta(q) = \alpha^*(U_\theta,q)$ to hold:

for all $q \in [1,\bar{q}]$, $\alpha_\theta(q) : \theta \mapsto \alpha_\theta(q)$ is measurable, and \hspace{1cm} (15)

for all $\theta \in [\underline{\theta},\bar{\theta}], \alpha_\theta(\cdot) : q \mapsto \alpha_\theta(q)$ is continuous on $q \in [1,\bar{q}]$. \hspace{1cm} (16)

It remains to show that these assumptions are indeed fulfilled. Equations (15) and (16) are fulfilled under the following very general conditions (see appendix for a formal proof):

for all $\theta \in [\underline{\theta},\bar{\theta}] U_\theta : [W-L,W] \rightarrow \mathbb{R},(x) \mapsto U_\theta(x)$ \hspace{1cm} (17)
is a strictly concave, increasing, and continuous utility function, and, for all

\[ x \in [W - L, W], \text{ the map } U_\theta : [\underline{\theta}, \bar{\theta}] \to \mathbb{R} \to U_\theta(x) \text{ is continuous.} \quad (18) \]

The conditions above generally hold for all important and commonly used utility functions, which are those of the HARA (Hyperbolic Absolute Risk Aversion) class—i.e., exponential, quadratic, and isoelastic utility functions. The conditions also hold for root functions, as can be shown by straightforward proof.

A further important application of this model framework is “kinked” utility functions, which are continuous but not differentiable at some discrete points. In this case, \( \alpha^*(U_\theta, q) \) is still continuous but not necessarily differentiable any more. However, our assumptions (17) and (18) also hold in this case since we mainly postulate continuity but not differentiability. Therefore, an optimal pricing strategy also exists.

For a given individual, the expected profit of the insurer amounts to

\[ \Pi_\theta(q) = \alpha^*(U_\theta, q)(q - 1)pL. \quad (19) \]

The aggregate profit of the insurer, given all risk-averse policyholders, is then given by

\[ \Pi(q) = \int_{\underline{\theta}}^{\bar{\theta}} \Pi_\theta(q)f(\theta)d\theta = (q - 1)pL\int_{\underline{\theta}}^{\bar{\theta}} \alpha^*(U_\theta, q)f(\theta)d\theta. \quad (20) \]

Making the reasonable assumption that \( q \) lies in a compact interval \( [1, \bar{q}] \), our problem is to maximize a continuous function over a compact interval. Therefore, a profit-maximizing \( q \) always exists. We generalize the optimality conditions for the insurer first found by Schlesinger (1983: 75–78). We only need continuity in \( q \). Under this very general and weak assumption, we demonstrate the existence of an optimal premium maximizing the insurer’s expected profit. This result is particularly important since it includes existence of optimal insurance pricing for “kinked” utility functions. A formal proof is given in the appendix.

In summary, we find the expected profit function of the insurer to be continuous on a compact interval implying a general maximum. The insurer can thus always find an insurance premium for all potential policyholders that maximizes its expected profit. In particular, we offer general and sufficient assumptions to ensure the existence of optimal insurance pricing even if utility functions exhibit one or even more “kinks.”
To illustrate our results, consider the easier case of a differentiable function of individually optimal insurance coverage \( \alpha^* \). The optimality condition for (20) becomes

\[
q^* = 1 - \frac{\int_0^\theta \alpha^*(U_{\theta,q})f(\theta)d\theta}{\int_0^\theta \partial_\theta \alpha^*(U_{\theta,q}) (f(\theta)d\theta)}
\]

(21)

where the optimal \( q \) is such that elasticity of expected cost with respect to \( q \) is unitary (Schlesinger, 1983: 76).

**COMBINATION OF LOADINGS**

We also consider the general case of a combination of fixed and proportionate premium loadings. In this case, the insurer offers a premium of the structure \( P = \alpha q p L + m \). Here the fixed loading fee \( m \) presents some kind of entry cost into the insurance market. Generally, a customer will decide on the basis of his individual risk premium whether he enters or not. Once this entry cost has been paid, the insured chooses his individually optimal level of insurance coverage. The function of expected profit of the insurer is then given by

\[
\Pi(m,q) = \int_0^{\bar{\theta}} \alpha^*(U_{\theta,q})(q - 1)p L f(\theta)d\theta + \{1 - F(\theta_0(m,q))\} \cdot m
\]

(22)

with \( \theta_0(m,q) = \inf \{ \theta \in [\underline{\theta},\bar{\theta}] : \alpha^* U_{\theta,q} \cdot 1_{[r(\theta) \geq m]} > 0 \} \)

(23)

\( \Pi(m,q) \) is a composition of continuous functions, and thus is itself continuous on the compact set \( [\underline{\theta},\bar{\theta}] \times [1,q] \). As a consequence, the function has a maximum. Therefore, in the general case of a combination of both potential premium loadings, the resulting profit function of the insurer (22) for root utility may be depicted as in Figure 1. The maximum is obtained at approximately \( m = 1.50 \) and \( q = 1.14 \).

Finally, we may consider how the expected profit of the insurer changes with a variation of the premium loading—i.e., price structure of the insurer. Generally, the expected profit of the insurer will change when the premium changes from a fixed premium loading to a proportionate one.
or even to a combination of both. In the case of a combination of loadings, the expected profit of the insurer is always at least as high as in each single case—that is, when each pricing rule is applied separately. This follows directly from

\[
\{1 - F(\theta_0(p,q))\} \cdot p \geq 0
\]

and

\[
\max \Pi(m,q) \geq \max \Pi(m,1) = \max (1 - F(\theta_0(m,1))) \cdot m.
\]

As a result, the general premium structure incorporates the potential to extract expected profit from both fixed loading fee and proportionate loading factor. Therefore, this general pricing strategy seems to be a powerful instrument.

The potential improvement in expected profit for the monopolist is depicted for different density functions in Table 1. In particular, the table illustrates the percentage increase in expected profit when the insurer switches from a simple fixed or proportionate loading scheme to a combination of loadings. When we compare the columns in the table, we find

**Fig. 1.** Expected profit under combination of loadings for \( W = 100, L = 50, \) and \( p = 0.25. \)
that the resulting percentage increase in expected profit of the insurer is higher in case of an extension of the pricing strategy from a solely proportionate loading strategy to the general combined premium loading scheme. In contrast, the extension to a general combined premium loading scheme given solely a fixed loading strategy involves less additional expected profit for the insurer. Therefore, an insurer using a pricing strategy with a proportionate premium loading (only) might be worse off compared to an insurer using a pricing strategy with a fixed loading fee (only). This result is not due to our exemplified classes of utility functions above, but is still valid for other utility functions in the HARA class and other distribution of types, such as triangular, normal (with different expected value and variance) or gamma (with different expected value and variance) distributions. The results hold true not only for “normal” utility functions but also for “kinked” utility. The influence of utility functions and the actual distribution of underlying risk preferences seems to be of minor importance. Therefore, we may conclude that for commonly used utility and density functions, our results seem to be robust.

**CONCLUSION**

In a given population of potential customers, consumers with higher risk aversion have a higher willingness to pay for insurance than consumers with less risk aversion. An insurer may want to exploit these differences in order to maximize expected underwriting profit. We analyze hidden heterogeneity in risk preferences in an insurance market with a monopoly provider. While the insurer may easily determine the risk class a customer belongs to, the customers’ risk preferences within each risk group cannot be determined in an *ex ante* sense.
In our framework, we build upon earlier classics and formulate an approach for continuous utility to derive the optimal price for an insurer who cannot distinguish among individual types of customers. In a model exhibiting incomplete information (i.e., in a setting where the probability distribution of risk preferences is known to the insurer, but not individual types), we derive the resulting expected underwriting profit function of the insurer and derive optimal pricing strategies. We show that an optimal price always exists. This price is also optimal from a Nash equilibrium viewpoint since all consumers choose their most preferred contract from a given menu. The insurer offers its expected profit maximizing price. Hence both insurer and consumers choose an optimal strategy given the other agents’ actions. Conditions for optimal pricing are mainly continuity assumptions of the utility function. These conditions are fulfilled for utility functions of the HARA class and “kinked” utility functions.

It is of interest to compare the different pricing strategies of the insurer and to discuss potential gains in expected profit when the insurer extends the premium structure to the general case involving both a proportionate and a fixed loading. Interestingly, in the case of commonly used density and utility functions, we find that given a pricing strategy with a proportionate loading only, the insurer might be worse off compared to a strategy with fixed loadings. Since the combined premium structure incorporates the potential to extract expected profit from both fixed loading fee and proportionate loading factor, this general pricing strategy is superior when compared to the simple well-known premium strategies discussed in the literature.

Finally, it should be mentioned that our model refers to a continuum of different customer types within a given risk class: we assume customer heterogeneity in only one characteristic. This might seem restrictive. If, however, customers differ in both their risk-aversion and risk of loss, a two-dimensional asymmetric information problem results. This makes screening consumers more complicated. It may change the nature of optimal outcomes in significant ways and may therefore even matter in a competitive insurance market. Smart (2000) studies a competitive insurance market in which insurance customers differ in two dimensions: they exhibit either high or low risk aversion, and, at the same time, either high or low accident probability. He investigates existence and properties of a possible equilibrium and shows that preferences (i.e., indifference curves) may be double-crossing (rather than single-crossing). A pure-strategy subgame perfect equilibrium may be unique, if it exists. If an equilibrium exists, it may be either separating or pooling involving cross-subsidization among types of customers. Hence, the qualitative features of well-known equilibria in canonical insurance models may not be preserved in more complex
environments where a second dimension of agents’ characteristics is added to the analysis.

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REFERENCES


**APPENDIX A**

In the following, we show the continuity of \( \Pi(q) = \int_{\theta}^{\bar{\theta}} \Pi_{\theta}(q)f(\theta)d\theta \) in \( q \). For that purpose, assume \( q_n \to q \) to be any sequence with \( q_n,q \in [1,\bar{q}] \).

Due to measurability of \( \alpha^*(U_{\theta},q) \) in \( \theta \), the functions \( \Pi_{\theta}(q_n)f(\theta) \) and \( \Pi_{\theta}(q)f(\theta) \) are measurable in \( \theta \). Resulting from continuity of \( \alpha^*(U_{\theta},q) \) in \( q \), we obtain

\[
\Pi_{\theta}(q_n)f(\theta) = \alpha^*(U_{\theta},q_n)(q_n - 1)pLf(\theta) \to \alpha^*(U_{\theta},q)(q - 1)pLf(\theta) = \Pi_{\theta}(q)f(\theta).
\]

In addition, we have

\[
\left| \Pi_{\theta}(q_n)f(\theta) \right| = \left| \alpha^*(U_{\theta},q_n)(q_n - 1)pLf(\theta) \right| \leq 1\bar{q}pL_{\max} f(\theta):g
\]

for integrable function \( g \). Following the Dominated Convergence Theorem [Browder, 1996: 230] we may interchange integration and lims so that

\[
\int_{\theta}^{\bar{\theta}} \Pi_{\theta}(q)f(\theta)d\theta = \lim_{n \to \infty} \int_{\theta}^{\bar{\theta}} \Pi_{\theta}(q_n)f(\theta)d\theta.
\]

As a consequence, we obtain \( \Pi(q_n) \to \Pi(q) \). This proves the continuity of \( \Pi \) in \( q \).
APPENDIX B

Let $g_\theta:[0,1] \to \mathbb{R}, g_\theta(\alpha) = EU_\theta(W - \alpha \pi - (1 - \alpha)X)$.

**Assertion 1:**

For each $\theta \in [\underline{\theta}, \bar{\theta}]$, $g_\theta$ has a unique maximum $\alpha^*(\theta)$.

**Proof.** By condition (17), the map $h_\theta(\alpha) = U_\theta(W - \alpha \pi - (1 - \alpha)X)$ is a continuous function on $[0,1]$ which is bounded by $U_\theta(W)$. By the Dominated Convergence Theorem, it follows that $g_\theta$ is continuous, as well. Hence, it attains its maximum. Moreover, $g_\theta$ is strictly concave: For $\alpha_1, \alpha_2 \in [0,1]$ and $\lambda \in [0,1]$ we have,

$$U_\theta(\lambda(W_0 - \alpha_1 \pi - (1 - \alpha_1)X) + (1 - \lambda)(W_0 - \alpha_2 \pi - (1 - \alpha_2)X)) \geq \lambda U_\theta(W - \alpha_1 \pi - (1 - \alpha_1)X) + (1 - \lambda)U_\theta(W - \alpha_2 \pi - (1 - \alpha_2)X)$$

with strict inequality if $\alpha_1 \pi + (1 - \alpha_1)X \neq \alpha_2 \pi + (1 - \alpha_2)X$ (i.e. $X \neq \pi$), which we assume to be positive with probability $>0$.

Hence, we have

$$g_\theta(\lambda \alpha_1 + (1 - \lambda)\alpha_2) > \lambda g_\theta(\alpha_1) + (1 - \lambda)g_\theta(\alpha_2).$$

As a result, the maximum is unique.

**Assertion 2:**

The map $\alpha^*:[\underline{\theta}, \bar{\theta}] \to [0,1]$ is continuous (and thus measurable).

**Proof.**

Remark: Since all $U_\theta$ are continuous and defined upon a compact interval, uniform continuity follows:

$$\sup_{W - L \leq X \leq W} |U_{\theta_n}(x) - U_\theta(x)| \to 0 \text{ for all } \theta_n \to \theta.$$
Let \( \theta_n \) be a sequence converging to \( \theta_0 \). For all \( \delta > 0 \), the continuous function \( g_{\theta_0} \) attains its maximum \( M \) on the compact set \([0,1] \setminus (\alpha^*(\theta_0) - \delta, \alpha^*(\theta_0) + \delta)\), and, by assertion 1, this maximum \( M \) is strictly smaller than \( g_{\theta_0}(\alpha^*(\theta_0)) \). Next note that the map \( \theta \mapsto g_\theta \) is uniformly continuous by the uniform continuity of \( \theta \mapsto U_\theta \). Hence, there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \)

\[
\sup_{\alpha \in [0,1]} \left| g_{\theta_n}(\alpha) - g_{\theta_0}(\alpha) \right| \leq \frac{g_{\theta_0}(\alpha^*(\theta_0)) - M}{3} = : \eta.
\]

It follows that

\[
g_{\theta_n}(\alpha^*(\theta_0)) \geq g_{\theta_0}(\alpha^*(\theta_0)) - \eta
\]

and for all \( \alpha \notin (\alpha^*(\theta_0) - \delta, \alpha^*(\theta_0) + \delta) \) we have

\[
g_{\theta_n}(\alpha) \leq g_{\theta_0}(\alpha) + \eta \leq M + \eta = \frac{2M + g_{\theta_0}(\alpha^*(\theta_0))}{3} < \frac{M + 2g_{\theta_0}(\alpha^*(\theta_0))}{3} = g_{\theta_0}(\alpha^*(\theta_0)) - \eta \leq g_{\theta_n}(\alpha^*(\theta_0)).
\]

Thus the maximum of \( g_{\theta_n} \) must belong to the interval \((\alpha^*(\theta_0) - \delta, \alpha^*(\theta_0) + \delta)\), that is, we proved that

\[
\forall \delta > 0 \exists n_0 \forall n \geq n_0 \left| \alpha^*(\theta_n) - \alpha^*(\theta_0) \right| < \delta,
\]

which is the asserted continuity.

Now we assume that the premium depends upon some extra parameter \( q \in J \) for some interval \( J \). We assume that the map \( q \mapsto \pi_q \) is continuous. Note that in our case, this map \( q \mapsto \pi_q = qpL \) is continuous. We denote similarly as above
\[ g_{\theta}(\alpha,q): = EU_{\theta}(W - \alpha \pi_q - (1 - \alpha)X) \]

and

\[ \alpha^*(\theta,q): = \arg \max_{\alpha \in [0,1]} g_{\theta}(\alpha,q). \]

**Assertion 3:**
The map \( \alpha^*(\theta,\cdot): J \to [0,1] \) is continuous for all \([\theta,\bar{\theta}]\).

**Proof.** The proof is completely analogous to the proof of assertion 2.