Interest-Bearing Surplus Model with Liquid Reserves

Kristina P. Sendova¹ and Yanyan Zang²

Abstract: We consider a ruin model where the surplus process of an insurance company is constructed so that part of the current surplus is kept available at all times and the remaining part is invested. The former portion of the capital is called “liquid reserves.” In this paper, we study the expected discounted penalty function at ruin. First, we derive an integro-differential equation satisfied by the Gerber-Shiu function. Second, we apply Laplace transforms to the equation and reduce it to a first order linear differential equation for the function in question. Finally, we find an explicit form of the Gerber-Shiu function by considering exponential claims. [Key words: Gerber-Shiu function, interest, liquid reserves.]

INTRODUCTION

The Office of Superintendent of Financial Institutions (OSFI) in Canada has released Minimum Continuing Capital and Surplus Requirements (MCCSR), which are in place to protect policyholders by ensuring that insurance companies maintain adequate capital levels while the remaining surplus of the insurer may be invested in a competitive global marketplace. In this paper, we intend to model these requirements. However, it is not possible to implement MCCSR in their full detail due to their high complexity. Instead, we apply some basic ideas from MCCSR to construct the surplus process. Previously, this was achieved in two papers by Cai et al.

¹Department of Statistical and Actuarial Sciences, University of Western Ontario, ksendova@stats.uwo.ca
²Department of Statistical and Actuarial Sciences, University of Western Ontario, yzang4@uwo.ca

The authors wish to thank an anonymous referee for numerous valuable suggestions that undoubtedly improved the clarity of the paper. Support from a grant from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged by the first author.
(2009a) and Cai et al. (2009b), who chose a threshold-type model. More specifically, the authors assume that the portion of the surplus that is below a preset level is liquid and the amount in excess of this level is invested under a deterministic interest rate. In the present paper, instead of implementing a threshold, we prefer to invest a percentage of the current surplus of the insurer. This approach is more in line with the MCCSR.

Our objective is to study the expected value of the discounted penalty function at ruin under a specific surplus process. This function is also known as the Gerber-Shiu discounted penalty function, which we define by

\[ m(u) = E\{e^{-\delta T} w(U(T-),|U(T)|)I(T < \infty) | U(0) = u \}, \quad u \geq 0. \]

It was first introduced by Gerber and Shiu (1998) for analyzing the classical compound Poisson model and was subsequently studied under numerous other ruin theory models.

Here \( T \) is the time of ruin, which is denoted by \( T = \inf\{t | U(t) < 0\} \), \( \delta \geq 0 \) is the force of interest, and \( w(x_1, x_2), x_1 \geq 0, x_2 > 0 \), is a nonnegative function of the surplus immediately before ruin (\( U(T-) \) above) and the deficit at ruin (\( |U(T)| \) above). The function is known as penalty function. Also, \( I(E) \) is the indicator function of an event \( E \). The indicator assigns a value of one when the event occurs and zero when the event does not occur.

Initially, the function was intended as a tool for analyzing the expected discounted penalty as a function of the surplus \( x_1 \) and the deficit \( x_2 \). It has, though, a much broader meaning and serves to recover a number of quantities of special interest in ruin theory. These include the probability of ultimate ruin, the Laplace transform of the time of ruin, the joint and marginal distributions and moments of the surplus immediately before ruin, and the deficit at ruin. The following list provides more detail how these quantities are obtained:

- probability of ruin: \( \delta = 0, w(x_1, x_2) = 1 \) for all \( x_1 \geq 0, x_2 > 0 \);
- (defective) joint and marginal moments of the surplus and deficit: \( \delta = 0, w(x_1, x_2) = x_1^k x_2^l, k, l \) nonnegative integers;
- (defective) discounted moments of the deficit: \( w(x_1, x_2) = x_2^l \);
- joint (defective) distribution of the surplus and deficit: \( \delta = 0, w(x_1, x_2) = I(x_1 \leq x)I(x_2 \leq y) = I(x_1 \leq x, x_2 \leq y) \) for all \( x_1 \geq 0, x_2 > 0 \);
- marginal (defective) distributions are obtained by letting either \( x \to \infty \) or \( y \to \infty \) in the above;
(defective) distribution of the claim causing ruin: $\delta = 0, w(x_1, x_2) = I(x_1 + x_2 \leq z)$ for all $x_1 \geq 0, x_2 > 0$;

- trivariate Laplace transform of the time of ruin, the deficit, and the surplus: $w(x_1, x_2) = e^{-sx_1-x_2}$; the marginal transforms are derived by setting any two of $\delta, s, z$ to 0.

Researchers attempt to express quantities of interest, such as some of those listed above, in terms of known quantities. A potential approach is to find an explicit expression for the Gerber-Shiu function and then investigate its particular cases. In several instances, it is not possible to obtain such an expression. Numerical methods might then be useful. Alternatively, specific cases of the expected discounted penalty function might be easier to analyze.

In this paper, we complement the work of Cai and Dickson (2002), Cai (2007), Yang et al. (2008), Cai et al. (2009a), and Cai et al. (2009b) where interest is also incorporated into the particular surplus processes. This article is structured as follows. In Section 2, we introduce the model we are interested in studying. Then, in Section 3, we derive an integro-differential equation for the Gerber-Shiu discounted penalty function and apply Laplace transforms to analyze this function in Section 4. We then try to obtain some explicit expressions when claim sizes are exponentially distributed, in Section 5. In contrast to previous articles considering ruin models with interest, we manage to keep all parameters in the model without fixing them to a particular value.

**MODEL DESCRIPTION**

Assume that the claim amounts $\{Y_1, Y_2, Y_3, \ldots\}$ are independent and identically distributed (i.i.d.) positive random variables with common cumulative distribution function (c.d.f.) $F(y), y > 0$, with $F(0) = 0$ and Laplace-Stieltjes transform $\tilde{f}(s) = \int_0^\infty e^{-sy} dF(y)$. Let the number-of-claims process $\{N(t) \mid t \geq 0\}$ be a homogeneous Poisson process with rate $\lambda > 0$, which is independent from $\{Y_1, Y_2, Y_3, \ldots\}$, and $\{V_1, V_2, V_3, \ldots\}$ be the times of respective claim occurrences. We suppose that the interclaim times $\{V_1, V_2 - V_1, V_3 - V_2, \ldots\}$, which are i.i.d. exponential random variables with mean $1/\lambda$, are also independent from the claim amounts. We also define
the aggregate-claims process \( S(t) = \sum_{i=1}^{N(t)} Y_i \mid t \geq 0 \) to be the sum of all
claims up to time \( t \), with the understanding that \( S(t) = 0 \) if \( N(t) = 0 \). Finally,
we assume the initial surplus to be \( u \geq 0 \) and the constant premium rate to
be \( c > \lambda \mathbb{E}[Y_1] \).

We intend to keep part of the difference between premiums and claims,
\( \alpha \in [0,1] \), available at all times and invest the remaining part with force of
interest \( r > 0 \). The surplus process \( \{U(t) \mid t \geq 0\} \) is then defined as

\[
U(t) = \alpha [ct - S(t)] + e^{rt} \left\{ u + (1 - \alpha) \left[ c \int_0^t e^{-rs} \, ds - \sum_{j=1}^{N(t)} e^{-rV_j} Y_j \right] \right\}. \tag{2.1}
\]

Observe that the force of interest \( r \) depends on the choice of investment
of the insurance company. The risk that is allowed by the MCCSR depends
on the type of insurance product and the rating that the company wants to
maintain. As for the force of interest \( \delta \) that appears in the Gerber-Shiu
function, it is set when and if the company bankrupts. Also, to obtain some
quantities of interest as special cases of the Gerber-Shiu function one needs
to fix \( \delta \) to a particular value or treat it as a variable of a Laplace transform.
It is then mathematically convenient to have the forces of interest \( r \) and \( \delta \)
as two separate parameters.

**INTEGRO-DIFFERENTIAL EQUATION FOR THE GERBER-SHIU FUNCTION**

In this section we derive an equation satisfied by the Gerber-Shiu
function. Since the function of interest is found under both the integral sign
and the derivative sign, the identity that we obtain is called *integro-differ-
ential equation*. As it is, numerical approaches may be used for solving it. In
our later studies, though, we consider two other approaches to deducing
the expected discounted penalty function.

**Theorem 3.1** *Under the surplus process \( U(t) \), which is governed by identity
(2.1), the Gerber-Shiu function \( m \) satisfies the following integro-differential
equation*
\begin{equation}
m'(u) = \frac{\lambda + \delta}{c + ru} m(u) - \frac{\lambda}{c + ru} \left[ \int_{0}^{u} m(u - y) dF(y) + \int_{u}^{\infty} w(u, y-u) dF(y) \right], \quad (3.1)
\end{equation}

\(u \geq 0.\)

Proof. Assuming that the time of the first claim is \(t\) and its amount is \(y\), the surplus \(U(t)\) may be presented as

\[
U(t) = \alpha c t + e^{rt} u + \frac{(1-\alpha)c}{r} (e^{rt} - 1) - y
\]

\[= \sigma(t; u) - y,
\]

where

\[
\sigma(t; u) = \alpha c t + e^{rt} u + \frac{(1-\alpha)c}{r} (e^{rt} - 1).
\]

Let \(\Delta > 0\) be a sufficiently small number and consider the interval \([0, \Delta)\). Denote by \(m(u \mid E)\) the conditional expectation

\[
E \{ e^{-\delta T} w(U(T-), U(T)) \mid I(T < \infty), U(0) = u, E \}\]

where \(E\) is an event. Since the number of claims in \([0, \Delta)\) has a Poisson distribution with parameter \(\lambda \Delta\) and the interclaim times’ distribution is the memoryless exponential distribution, then conditioning on the number of claims in \([0, \Delta)\) and applying the Total Probability Theorem, we may rewrite the Gerber-Shiu discounted penalty function as

\[
m(u) = (1 - \lambda \Delta) e^{-\delta \Delta} m(u \mid \text{no claims in } [0, \Delta)) + \lambda \Delta e^{-\delta \Delta} m(u \mid \text{one claim in } [0, \Delta)) + o(\Delta).
\]

In the case when there are no claims in \([0, \Delta)\), the surplus at time \(\Delta\) is \(\sigma(\Delta; u)\). When one claim of amount \(y\) occurs in \([0, \Delta)\) there are two possible scenarios: this claim is smaller than the accumulated capital and the surplus process starts over with a new initial surplus of \(\sigma(\Delta; u) - y\), or the claim amount leads to ruin with penalty \(w[\sigma(\Delta; u), \sigma(\Delta; u) - y]\). We may then apply the Total Probability Theorem for all possible amounts \(y\) and obtain
\[ m(u) = (1 - \lambda \Delta) e^{-\delta \Delta} m(\sigma(\Delta; u)) + \] (3.2)

\[ \lambda \Delta e^{-\delta \Delta} \left\{ \int_0^{\sigma(\Delta; u)} m(\sigma(\Delta; u) - y) dF(y) + \int_0^{\infty} w(\sigma(\Delta; u), \sigma(\Delta; u) - y) dF(y) \right\} + o(\Delta). \]

By employing Taylor’s expansion, \( m[\sigma(\Delta; u)] \) is presented as

\[ m[\sigma(\Delta; u)] = m(u) + m'(u)[\sigma(\Delta; u) - u] + \sum_{n=2}^{\infty} \frac{m^{(n)}(u)}{n!} [\sigma(\Delta; u) - u]^n. \]

Then, inserting this equation into (3.2) yields

\[ m(u) = (1 - \lambda \Delta) e^{-\delta \Delta} \left\{ m(u) + m'(u)[\sigma(\Delta; u) - u] + \sum_{n=2}^{\infty} \frac{m^{(n)}(u)}{n!} [\sigma(\Delta; u) - u]^n \right\} \]

\[ + \lambda \Delta e^{-\delta \Delta} \left\{ \int_0^{\sigma(\Delta; u)} m(\sigma(\Delta; u) - y) dF(y) + \int_0^{\infty} w(\sigma(\Delta; u), \sigma(\Delta; u) - y) dF(y) \right\} + o(\Delta). \]

We move all the terms to the left-hand side and divide by \( \Delta \)

\[ 0 = \frac{1 - e^{-\delta \Delta}}{\Delta} m(u) - \frac{(1 - \lambda \Delta) e^{-\delta \Delta}}{\Delta} [\sigma(\Delta; u) - u] m'(u) \]

\[ - \frac{(1 - \lambda \Delta) e^{-\delta \Delta}}{\Delta} \sum_{n=2}^{\infty} \frac{m^{(n)}(u)}{n!} [\sigma(\Delta; u) - u]^n \]

\[ - \frac{\lambda e^{-\delta \Delta}}{\Delta} \left\{ \int_0^{\sigma(\Delta; u)} m(\sigma(\Delta; u) - y) dF(y) + \int_0^{\infty} w(\sigma(\Delta; u), \sigma(\Delta; u) - y) dF(y) \right\} \]

\[ - o(\Delta). \]
We apply L'Hôpital's rule to the above equation when \( \Delta \to 0 \) and the terms involving \( \Delta \) become

\[
\lim_{\Delta \to 0} \frac{1 - e^{-\delta \Delta + \lambda \Delta e^{-\delta \Delta}}}{\Delta} = \lim_{\Delta \to 0} \left( \delta e^{-\delta \Delta + \lambda \Delta e^{-\delta \Delta}} - \lambda \delta e^{-\delta \Delta} \right) = \lambda + \delta ,
\]

\[
\lim_{\Delta \to 0} \sigma(\Delta; u) = \lim_{\Delta \to 0} \left[ \alpha c \Delta + u e^{r \Delta} + \frac{(1 - \alpha) c}{r} (e^{r \Delta} - 1) \right] = u ,
\]

\[
\lim_{\Delta \to 0} \frac{\sigma(\Delta; u) - u}{\Delta} = \lim_{\Delta \to 0} \left[ \frac{\alpha c \Delta + u e^{r \Delta} + \frac{(1 - \alpha) c}{r} (e^{r \Delta} - 1) - u}{\Delta} \right] = \left\{ \begin{array}{l}
\alpha c + ure^{r \Delta} + (1 - \alpha) ce^{r \Delta} , \quad n = 1 \\
0 , \quad n = 2 , 3 , 4 \ldots
\end{array} \right.
\]

\[
\lim_{\Delta \to 0} \frac{\sigma(\Delta)}{\Delta} = 0 .
\]

Hence, equation (3.2) yields

\[
(\lambda + \delta)m(u) - (c + ru)m'(u) - \int_{0}^{u} m(u - y)dF(y) - \int_{u}^{\infty} w(u, y - u)dF(y) = 0
\]

which is the required integro-differential equation for the Gerber-Shiu function.

Observe that the integro-differential equation satisfied by the Gerber-Shiu function does not depend on the choice of \( \alpha \), where \( \alpha \) was introduced as the rate of the difference between premiums and claims to be kept available at all times. This is a notable difference from the ruin models
proposed by Cai et al. (2009a) and Cai et al. (2009b), where all expressions depend on the threshold level.

Cai and Dickson (2002) discuss the case when $\alpha$ is equal to 0, which is interpreted as investing the whole amount of surplus with certain force of interest, and the integro-differential equation (2.2) derived in their paper is exactly the same as the equation obtained in Theorem 3.1.

In contrast to previous ruin models that lead to integro-differential equations satisfied by the discounted penalty function (see, e.g., Lin et al., 2003; Mitric et al., 2010), identity (3.1) deduced in Theorem 3.1 is linear but with non-constant coefficients. It is exactly the term $c + ru$ in (3.3) that leads to complications. To obtain an explicit result, one needs to impose additional assumptions on the force of interest $\delta$, the penalty function $w$, or the claim-size distribution. For example, the most general explicit solution to (3.1) appears in the paper by Cai and Dickson (2002), where $\delta$ is set to zero. (This result is also implemented in the models considered by Cai et al., 2009a, and Cai et al., 2009b.) Less general is the result of Theorem 5.1 in Cai (2007), where along with the assumption $\delta = 0$ the author considers exponential claims. He then deduces an explicit expression of the Gerber-Shiu function. The least general result appears in Yang et al. (2008), who make the previous two assumptions and suppose further that the penalty function depends only on the deficit at ruin.

In Section 4 of this paper, we propose an alternative expression to the integro-differential equation (3.1), which is the first order non-homogeneous differential equation (4.1) satisfied by the Laplace transform of the expected discounted penalty function. Although this type of equation is well studied in the area of ordinary differential equations and may be solved explicitly, identity (4.1) poses some difficulties. As a result, Theorem 4.1 may rather provide an alternative equation to be analyzed numerically.

Another contribution to the existing literature is the explicit form of the Gerber-Shiu function provided by Theorem 5.2 in Section 5, where claims are assumed to follow an exponential distribution but no other restrictions are imposed to the model parameters.

**LAPLACE TRANSFORM OF THE GERBER-SHIU FUNCTION**

Cai and Dickson (2002) present equation (3.1) as a Volterra-type integral equation in their attempt to solve it. As a consequence, for the explicit solution they need to specify the value of the Gerber-Shiu function with no initial surplus. They achieve this in the discount free case, i.e., when $\delta = 0$.

In this section we attempt to study the behavior of the Gerber-Shiu function through its Laplace transform. As long as continuous integrable
functions are concerned, the inverse of their Laplace transform is unique (see Theorem 2-1 in Spiegel, 1965). As a result, obtaining an explicit expression for the Laplace transform of such a function yields an explicit expression for the function being transformed, as long as a way of inverting the Laplace transform is found. This is why Laplace transforms are often characterized for ruin models discussed in the actuarial literature. Researchers then either try to invert the Laplace transform and deduce an expression of the expected discounted penalty function, or apply a numerical method for the inversion. As it will be seen later, neither of these two approaches is possible under the model studied in this paper. Thus, one needs to resort to numerical methods available for non-homogeneous linear ordinary differential equations.

Denote by \( \tilde{m}(s) = \int_{0}^{\infty} e^{-su} m(u) du \) the Laplace transform of the Gerber-Shiu function. We also define \( \zeta(u) = \int_{u}^{\infty} w(y, u - y) dF(y) \) and

\[ \tilde{\zeta}(s) = \int_{0}^{\infty} e^{-su} \zeta(u) du. \]

**Theorem 4.1** The Laplace transform of the Gerber-Shiu function satisfies

\[ \tilde{m}'(s) = g(s) \tilde{m}(s) + h(s) \quad \text{(4.1)} \]

where

\[ g(s) = \frac{1}{rs} \left[ cs - r - \lambda - \delta + \lambda \tilde{f}(s) \right] \]

and

\[ h(s) = \frac{\lambda}{rs} \tilde{\zeta}(s) - \frac{c}{rs} m(0). \]

**Proof.** From Theorem 3.1, we have

\[ m'(u) = \frac{\lambda + \delta}{c + ru} m(u) - \frac{\lambda}{c + ru} \left[ \int_{0}^{u} m(u - y) dF(y + \zeta(u)) \right], \quad u \geq 0. \]

Applying Laplace-Stieltjes transforms to the above equation and rearranging it, we obtain
\[
\int_0^\infty e^{-s\mu}(c + ru)m'(u)du = \\
(\lambda + \delta)\int_0^\infty e^{-s\mu}m(u)du - \lambda\int_0^\infty s\int_0^u e^{-s\mu}m(u - y)dF(y)du - \lambda\int_0^\infty e^{-s\mu}\zeta(u)du.
\]

We then rewrite the above equation as
\[
c\int_0^\infty e^{-s\mu}m'(u)du + r\int_0^\infty e^{-s\mu}um'(u)du = \\
(\lambda + \delta)\tilde{m}(s) - \lambda\int_0^\infty \int_y^\infty e^{-s\mu}m(u - y)dudF(y) - \lambda\zeta(s).
\]

Performing integration by parts on the left-hand side, we have
\[
c\int_0^\infty e^{-s\mu}m'(u)du + r\int_0^\infty e^{-s\mu}um'(u)du = c\left[\int_0^\infty e^{-s\mu}m(u)du + \int_0^\infty se^{-s\mu}m(u)du\right] +
\]
\[
- r\left[\left. e^{-s\mu}um(u)\right|_0^\infty - \int_0^\infty (e^{-s\mu} - sue^{-s\mu})m(u)du\right]
\]
\[
= c[s\tilde{m}(s) - m(0)] + r[- \tilde{m}(s) - s\tilde{m}'(s)]
\]
\[
= (cs - r)\tilde{m}(s) - cm(0) - rs\tilde{m}'(s).
\]

Then the equation becomes
\[
(\lambda + \delta)\tilde{m}(s) - \lambda\int_0^\infty e^{-s(t + y)}m(t)\Delta F(y) - \lambda\zeta(s)
\]
\[
= (\lambda + \delta)\tilde{m}(s) - \lambda\tilde{m}(s)f(s) - \lambda\zeta(s).
\]

Finally,
\[
\tilde{m}'(s) = \frac{1}{rs}[cs - r - \lambda - \delta + \lambda f(s)]\tilde{m}(s) + \frac{\lambda}{rs}\zeta(s) - c m(0),
\]
as needed.
Observe that in contrast to the classical compound Poisson model, $s = 0$ is not a root to the function $cs + \lambda \tilde{f}(s) - (\lambda + \delta + r)$ even when $\delta = 0$, unless $r = 0$ as well. Thus, it is sufficient to consider only the case $s > 0$.

Notice that the highest derivative of the Laplace transform $\tilde{m}$ in equation (4.1) is the first derivative. Also, it contains a term $h$ that does not involve $\tilde{m}$. This makes the equation non-homogeneous with respect to $\tilde{m}$. In other words, we have derived a linear non-homogeneous first order differential equation satisfied by the Laplace transform of the Gerber-Shiu function. We are able to solve this ordinary differential equation through standard techniques (see, for instance, in Petrovski, 1966, p. 21). Namely,

$$\tilde{m}(s) = m_0 \exp \left[ \int_{s_0}^{s} g(\xi) d\xi \right] + \int_{s_0}^{s} h(x) \exp \left[ \int_{\chi}^{s} g(\xi) d\xi \right] dx,$$

where $s_0 > 0$ is a constant such that $m_0 = \tilde{m}(s_0)$ is known. We already know that $\tilde{m}(s)$ converges to 0 when $s$ converges to $\infty$. This is why we let $s_0 = \infty$ and $m_0 = 0$. Next, we need to verify whether $\int_{s_0}^{s} g(\xi) d\xi$ converges or not when $s_0 \to \infty$.

$$\int_{s_0}^{s} g(\xi) d\xi = -\int_{s}^{\infty} g(\xi) d\xi$$

$$= -\int_{s}^{\infty} \frac{1}{r\xi} [c\xi - r - \lambda - \delta + \lambda \tilde{f}(\xi)] d\xi$$

$$= -\int_{s}^{\infty} \frac{c}{r\xi} - \frac{1}{r\xi} (r + \lambda + \delta) + \frac{\lambda}{r} \tilde{f}(\xi) d\xi$$

$$= -\left[ \frac{c\xi}{r} - \frac{r + \lambda + \delta}{r} \log \xi \right]_{s}^{\infty} - \frac{\lambda}{r} \int_{s}^{\infty} \frac{\tilde{f}(\xi)}{\xi} d\xi.$$

We find that the first term of the equation on the right-hand side converges to $-\infty$. Then we need to check how the second term behaves. We begin by applying Theorem 1-11 in Spiegel (1965).
Either of the two representations of the integral \( \int_s^\infty \frac{\tilde{f}(\xi)}{\xi} d\xi = \int_s^\infty e^{-yF(y)}dy \) are difficult to test for convergence unless the claim-size c.d.f. \( F \) is known. As long as our problem is concerned, though, it is sufficient to know that they converge either to a positive finite value or to infinity. As a result, the whole integral \( \int_s^\infty \frac{\tilde{f}(\xi)}{\xi} d\xi = \int_s^\infty e^{-sF(y)}dy \) converges to 0. Thus, the first term on the right-hand side of the equation (4.2), namely, \( m_0 \), is 0. Thus, equation (4.2) reduces to

\[ \tilde{m}(s) = \int_s^\infty h(x) \exp \left[ \int_x^s g(\xi) d\xi \right] dx . \] (4.3)

We now want to consider the second part of equation (4.2), namely, the right-hand side of (4.3). However, it is difficult to clarify its convergence properties because the integral becomes complex after expanding. Moreover, it involves \( m(0) \), which needs to be specified additionally. Cai and Dickson (2002) manage to do so only in the discount-free case when \( \delta = 0 \). Therefore, it seems that numerical approaches to solving equation (4.1) might be more appropriate.

**EXPOSENTIAL CLAIMS**

In this section, we derive an explicit expression for the Gerber-Shiu function when claim amounts are exponentially distributed with no other restrictions. It is noteworthy that the papers by Cai (2007), Yang et al. (2008), Cai et al. (2009a), and Cai et al. (2009b) all consider the case with exponential
claim amounts but restrict themselves to the discount-free case, i.e., when \( \delta = 0 \). Moreover, Yang et al. (2008) impose further that the penalty function depend solely on the deficit at ruin.

We proceed by restating the integro-differential equation for the Gerber-Shiu function as

\[
(c + ru)m'(u) - (\lambda + \delta)m(u) = -\lambda[A(m(u)) + \zeta(u)]
\]

(5.1)

where \( A(m(u)) = \int_0^u m(u - y)dF(y) \) and the claim amounts \( \{Y_1, Y_2, \ldots\} \) are independent, identically, and exponentially distributed with common p.d.f. \( f(y) = \beta e^{-\beta y} \), \( y > 0 \).

**Lemma 5.1** When claim sizes are exponentially distributed, then

\[
\frac{d}{du}A(m(u)) = \beta m(u) - \beta A(m(u)).
\]

**Proof.** By a change of variables we have,

\[
A(m(u)) = \int_0^u m(y)f(u - y)dy = \int_0^u m(y)\beta e^{-\beta(u - y)}dy.
\]

Then we differentiate the equation with respect to \( u \),

\[
\frac{d}{du}A(m(u)) = \beta m(u) + \int_0^u \frac{d}{du}[m(y)\beta e^{-\beta(u - y)}dy].
\]

\[
= \beta m(u) - \beta \int_0^u m(y)\beta e^{-\beta(u - y)}dy
\]

\[
= \beta m(u) - \beta A(m(u)),
\]

which completes the proof.

We now reduce the integro-differential equation (5.1) to a second order non-homogeneous differential equation. To this aim, we first differentiate equation (5.1) and it becomes

\[
(c + ru)m''(u) + (r + \lambda + \delta)m'(u) = \lambda \left[ \frac{d}{du}A(m(u)) + \zeta(u) \right].
\]

(5.2)

Then, we multiply equation (5.1) by \( \beta \),
\[
\beta(c + ru)m''(u) - \beta(\lambda + \delta)m(u) = -\lambda \beta[A(m(u)) + \zeta(u)].
\] (5.3)

By adding (5.2) and (5.3) together and employing Lemma 5.1, we obtain
\[
(c + ru)m''(u) + [r - \lambda - \delta + \beta(c + ru)]m'(u) - \beta \delta m(u) = \lambda [\zeta'(u) + \beta \zeta(u)],
\] (5.4)
as needed. We may now solve equation (5.4) explicitly.

**Theorem 5.2** When the claim amounts are exponentially distributed with mean \(1/\beta\), the Gerber-Shiu function has the form
\[
m(u) = [\kappa_1 + C_1(\beta u - \beta c/r)]y_1(\beta u - \beta c/r) + [\kappa_2 + C_2(\beta u - \beta c/r)]y_2(\beta u - \beta c/r),
\]
where \(\kappa_1\) and \(\kappa_2\) are arbitrary constants,
\[
y_1(x) = \begin{cases} 
\Phi\left(\frac{\lambda}{r}, 1 - \frac{\lambda + \delta}{r}; x\right), & \frac{\lambda + \delta}{r} \neq 1, 2, \ldots \\
\frac{\lambda + \delta}{x} \Phi\left(\frac{\lambda}{r}, 1 + \frac{\lambda + \delta}{r}; x\right), & \frac{\lambda + \delta}{r} = 1, 2, \ldots
\end{cases}
\] (5.5)
\[
y_2(x) = \begin{cases} 
\Psi\left(\frac{\lambda}{r}, 1 - \frac{\lambda + \delta}{r}; x\right), & \frac{\lambda + \delta}{r} \neq 1, 2, \ldots \\
\frac{\lambda + \delta}{x} \Psi\left(\frac{\lambda}{r}, 1 + \frac{\lambda + \delta}{r}; x\right), & \frac{\lambda + \delta}{r} = 1, 2, \ldots
\end{cases}
\] (5.6)
\[
\Phi(a, b; x) = 1 + \sum_{k=1}^{\infty} \frac{a^{(k)} x^k}{b^{(k)} k!}, \text{ and } a^{(k)} = a(a + 1)\ldots(a + k - 1),
\] (5.7)
\[
\Psi(a, b; x) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \Phi(a, b; x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} \Phi(a-b+1, 2-b; x),
\] (5.8)
\[
C_1 = k_1 \int_{0}^{x} y_1^{-2}(v) v^{-1+\frac{\lambda + \delta}{r}} e^{v} dv,
\] (5.9)
where $\kappa_1$ and $\kappa_2$ are also some arbitrary constants, and
\[
\eta(x) = \frac{\lambda}{r\beta} \left[ \zeta'(\frac{-1}{\beta}x - \frac{c}{r}) + \beta \zeta\left(-\frac{1}{\beta}x - \frac{c}{r}\right) \right].
\] (5.11)

Proof. We intend to make some substitutions in order to reduce equation (5.4) to a simpler form. Let $x$ be such that
\[
ru + c = -\frac{r}{\beta}x
\]
and $y(x)$ be such that
\[
m(u) = m\left(-\frac{1}{\beta}x - \frac{c}{r}\right) = y(x).
\]
As we know, $m'(u) = -\beta y(x)$ and $m''(u) = \beta^2 y''(x)$. Then equation (5.4) becomes
\[
-r\beta y''(x) - [r - \lambda - \delta - rx] \beta y'(x) - \beta \delta y(x) = \\
-\lambda \left[ \zeta'(\frac{-1}{\beta}x - \frac{c}{r}) + \beta \zeta\left(-\frac{1}{\beta}x - \frac{c}{r}\right) \right].
\]
By dividing the above equation by $-r\beta$, it reduces to
\[
xy''(x) + \left[1 - \frac{\lambda + \delta}{r} - x\right] y'(x) + \frac{\delta}{r} y(x) = h(x).
\] (5.12)
In order to solve the above non-homogeneous differential equation, we have to solve the associated homogeneous equation first, that is,
\[
xy''(x) + \left[1 - \frac{\lambda + \delta}{r} - x\right] y'(x) + \frac{\delta}{r} y(x) = 0.
\] (5.13)
The solution to equation (5.13) is provided in Polyanin and Zaitsev (2003), p. 220, and is described as the degenerate hypergeometric equation. It is stated as

\[ y_h(x) = \kappa_1 y_1(x) + \kappa_2 y_2(x), \]

where \( \kappa_1, \kappa_2 \) are some arbitrary constants, which may be found under boundary conditions, and \( y_1(x) \) and \( y_2(x) \) are specified by (5.5) and (5.6), respectively.

Polyanin and Zaitsev (2003) also state that the general solution to the non-homogeneous differential equation is the sum of the general solution to the corresponding homogeneous equation and any particular solution to the non-homogeneous equation. In our case, the particular form of the non-homogeneous equation (5.12) is

\[ y_p(x) = C_1(x)y_1(x) + C_2(x)y_2(x). \]

Now, we replace \( y(x) \) by \( y_p(x) \) in equation (5.12), deducing

\[
xy_1(x)C_1''(x) + \left[2xy_1'(x) + \left(1 - \frac{\lambda + \delta}{r} - x\right)y_1(x)\right]C_1'(x) + \\
xy_2(x)C_2''(x) + \left[2xy_2'(x) + \left(1 - \frac{\lambda + \delta}{r} - x\right)y_2(x)\right]C_2'(x) = \eta(x).
\]

Since we need only one particular solution to the non-homogeneous differential equation, we may find particular functions for \( C_1 \) and \( C_2 \) in order to make the above equation hold. One way of doing so without loss of generality is by assuming that

\[
xy_1(x)C_1''(x) + \left[2xy_1'(x) + \left(1 - \frac{\lambda + \delta}{r} - x\right)y_1(x)\right]C_1'(x) = 0
\]

and

\[
xy_2(x)C_2''(x) + \left[2xy_2'(x) + \left(1 - \frac{\lambda + \delta}{r} - x\right)y_2(x)\right]C_2'(x) = \eta(x).
\]

We are able to solve the above first order linear differential equations in \( C_1'(x) \) and \( C_2'(x) \).
where \( C_1'(x) \) passes through some point \((x_0, C_0)\), and \( l \) is some constant calculated from the integral. Thus, equation (5.9) represents the solution for \( C_1(x) \) and similarly, we may confirm the solution of \( C_2(x) \), which is defined in (5.10).

Therefore, the general solution to the non-homogeneous differential equation (5.12) has the form

\[
y(x) = y_h(x) + y_p(x),
\]

as needed.

**CONCLUSIONS**

In this paper we study the evolution of the surplus of an insurance company that invests a percentage \( 1 - \alpha \) of its current surplus into a risk-free asset that provides a return at a force of interest \( r \). The remaining percentage \( \alpha \) of the company’s surplus is allocated to the so-called “liquid reserves” that do not yield any return. Nevertheless, higher liquid reserves are tied to a higher credit rating. This is the incentive for the company to avoid setting \( \alpha \) to a very small value, or in other words, to invest almost entirely its surplus.

Under this setup, it is natural to ask what is an appropriate balance between investments and liquid reserves. The answer is provided by the value of \( \alpha \), which depends on the risk sensitivity of the company. More precisely, one needs to know how high a probability of ultimate ruin or a deficit at ruin the insurer is willing to tolerate. These two quantities and several other measures of the riskiness of the business are special cases of the expected discounted penalty function that is discussed in this paper.
Interestingly, it appears from our analysis that whether the insurance company keeps certain percentage $\alpha > 0$ of its current surplus as liquid reserves and invests the remainder, or the company invests the entire surplus ($\alpha = 0$), the expected discounted penalty function satisfies the same integro-differential equation. The dependence on $\alpha$ might be only relevant to the associated initial conditions. This in turn implies that any quantities of interest that may be deduced from the Gerber-Shiu function also depend on $\alpha$ only through some initial values. In particular, the probability of eventual ruin and the deficit at ruin, which are mentioned above, are related to $\alpha$ through these initial conditions.

Consequently, it becomes important to specify both the form of the Gerber-Shiu function and the related initial conditions. As long as the expected discounted penalty function is concerned, we propose two solution strategies. In regards to the initial conditions, Section 3 in Cai and Dickson (2002) provides a partial answer. Namely, when the force of interest $\delta$ is zero, $m(0)$ is specified and may serve as an initial condition.

Since there is no complete solution to the previously mentioned integro-differential equation satisfied by the Gerber-Shiu function in the actuarial literature, our first solution strategy is through its Laplace transforms. This leads to a simple linear ordinary differential equation that might be solved explicitly for particular cases of the penalty function as long as an initial condition is provided. In particular, this might be $m(0)$ when $\delta = 0$.

Our second solution strategy is related to the case when claims are exponentially distributed. We deduce an explicit form of the Gerber-Shiu function without imposing additional restrictions on the model’s parameters. The latter is a notable improvement over relevant results in the current actuarial literature.

Finally, our findings make us believe that although we provide an explicit expression of the Gerber-Shiu function when claims have exponential distribution, it is more practical to deduce the value of $m(u)$ for a specific initial capital $u$ numerically through the Laplace transform $\tilde{m}$ in equation (4.3).

REFERENCES


