Risk Modeling of Multi-year, Multi-line Reinsurance Using Copulas

Ping Wang

Abstract: Relationships between loss variables covered by multi-year, multi-line reinsurance are complicated in the sense that correlation may be present in two dimensions: from year to year and across business lines. While reinsurers may be able to choose and accept risk exposures from different lines of business to achieve risk diversification, it is unlikely that they will be able to completely eliminate the temporal dependence of annual loss experience throughout the period of coverage. Assuming temporal independence may likely result in an underestimation of the risk embedded in the contract. Using two independent lines of business, this paper applies copulas for the purpose of modeling the year-to-year dependence of annual loss experience of a multi-year, multi-line reinsurance agreement. We then simulate annual losses for separate lines of business by making random draws from different multivariate loss distributions and then analyze the characteristics of the resulting Total Loss variable upon which the contract payoff is determined.

As an illustrative example, we study the performance of a stop-loss reinsurance contract covering a primary insurer’s losses over a three-year period arising from workers compensation and commercial multiple perils coverage. This approach can be easily extended to address, simultaneously, dependencies among lines of business. [Key words: multi-year, multi-line, reinsurance, copula, risk modeling.]

1. INTRODUCTION

Reinsurance serves as a safety buffer through which primary insurers share with reinsurers the risks they underwrite from a wide variety of personal and commercial policyholders. Reinsurance protects insurers’ capital base against large deviations from expected losses, especially in the events of major catastrophes. By allowing primary insurers to manage their

1 The School of Risk Management, St. John’s University, New York, NY 10007, Wangp1@stjohns.edu
risks and capital in a more efficient way, reinsurance makes insurance more secure and less expensive. In 2010, direct premiums written by the global property-casualty insurance industry reached US$1,819 billion, of which over US$160 billion was ceded to reinsurance companies (Swiss Re, 2011a, 2011b). To ensure their ability to honor contractual obligations, reinsurers implement sophisticated risk management processes to identify, monitor and model the risks they accept from primary insurers. In addition, they carefully manage their assets and capital to be closely aligned with the risks that they have assumed. Easy market access to international risk transfer and free capital flow enable reinsurers to provide coverage for catastrophic events that would threaten the solvency of individual primary insurers.

Recent years have seen the emergence of the alternative risk transfer market, where risks can be transferred either to alternative carriers or through alternative products. The market for alternative carriers consists of self-insurance, captives, risk retention groups, or pools. Alternative risk transfer products, characterized by a mixture of risk transfer and risk finance, include finite risk reinsurance, run-off solutions, multi-trigger programs, structured finance, and multi-year, multi-line, property-casualty products. Alternative risk transfer products expand the limits of insurability and add to market capacity. See Culp (2006) for a detailed review of the traditional primary, reinsurance, and structured insurance markets. According to Swiss Re (2004), premiums from non-traditional products, mainly finite reinsurance, accounted for 10–15% of overall reinsurance premiums.

Risk modeling, the basis of risk management for reinsurers, involves different forms of qualitative and quantitative analyses for the purpose of risk identification and measurement. Alternative solutions may involve more risk finance than risk transfer. Nevertheless, reinsurers are still exposed to significant uncertainty in the timing and magnitude of contract payouts. The complicated designs of alternative risk transfer products present a challenge for risk modelers to forecast the amount and time of payout (see Swiss Re, 2004, for the descriptions of the ART instruments). This paper introduces a new method in modeling and analyzing the payout from a particular alternative risk product—namely, the multi-year, multi-line reinsurance contract.

With a multi-year, multi-line agreement, the reinsurer promises to pay only if the aggregated losses of the primary insurer arising from several business lines over a certain period exceed a fairly high threshold. In other terminology, the contracts are referred to as multi-year, multi-line stop-loss reinsurance products. If the multiple lines of business covered by the reinsurance contract are independent, the aggregate loss amount will be more predictable, thus reducing the uncertainty inherent in the estimation
process. The extended coverage period gives the primary insurer a chance to smooth loss experience by paying level premiums over multiple years.

For the reinsurer, it is important to forecast the cash flow of the contract payout, even if risk finance is the main purpose of the agreement. Modeling multi-year, multi-line reinsurance contracts is challenging because of the inherent complexity created by the correlations among annual loss experiences and across the multiple business lines covered. This paper applies copulas to the modeling and the analysis of such products.

A copula is a function that links univariate marginal distributions to the full multivariate distribution function. In the last decade, literature has been developed rapidly on the statistical properties and applications of copulas, particularly in the enterprise risk management literature (for example, Frees and Valdez (1998), Nelsen (1999), and Embrechts, Lindskog, and McNeil (2001)). Frees and Wang (2005, 2006) introduce applications of copulas in premium rating of property-casualty insurance. Venter et al. (2007) review generalizations of t-copulas and introduce a number of possible applications. Recently more applications of copulas to insurance and reinsurance operations have been published. Eling and Toplek (2009) employ copula models in a dynamic financial analysis of non-life insurers to acquire a better understanding of the effects of nonlinear dependencies on the risk and return profile. Applying a hierarchical model to individual automobile insurance policies, Frees et al. (2009) examine the predictive loss distribution for both the insurer and reinsurer under several reinsurance treaties. Unlike Frees et al. (2009), which works with personal lines data at the individual policy level, this paper models a multi-year, multi-line reinsurance contract utilizing commercial lines at the company level.

For multi-year, multi-line reinsurance agreements, dependence may be present in two dimensions: across years and across business lines. In this paper we use a copula framework to directly model the dependencies of loss experiences from year to year within individual lines of business while assuming independence across the multiple lines covered by the contract. This is realistic since the primary insurer and reinsurer both have the ability to cover business lines that are less dependent on each other. However, they generally have much less control over the temporal dependency of the loss experience from year to year. After fitting multivariate distributions to the data, we simulate the multivariate distribution of future annual loss variables to analyze the contract payout. In fact, the approach described in this paper can be easily extended to address both temporal dependencies and dependencies among lines of business.

Within the remainder of the paper, Section 2 offers a description of the basic stochastic model, including the marginal distributions and the copula for modeling dependencies over time. More mathematical and dataset
details are available in the Appendices. Section 3 introduces the data set and shows how to apply the framework described in Section 2. Section 4 applies simulation in predicting and analyzing the potential payout of a multi-year, multi-line reinsurance contract. Section 5 provides a summary and concluding remarks.

2. MODELING TIME-DEPENDENCE USING COPULAS

This section lays out the idea of applying copulas to modeling time-dependency. The process includes three parts: (i) fitting the marginal distributions using a parametric framework; (ii) connecting the marginal distributions via a copula; and (iii) predicting future payouts of the reinsurance contract using simulation.

Consider a reinsurance contract that covers two lines of insurance over a three-year period. Assume that the payout of the reinsurance contract is dependent on the multivariate distribution of \{X_t, X_2, Y_t, Y_2\}, where \(X_t\) and \(Y_t\) denote loss variables of the two lines of business covered and the subscript \(t\) is the index of time in years. We assume independence between variables \(X_t\) and \(Y_t\) while employing copulas to model the “from year to year” dependencies of the loss experience from individual lines separately.

Step 1. Fitting marginal distributions

The first step we employ is standard to actuaries and can be found in many actuarial textbooks (for example, Klugman et al. (2008)). We fit a variety of parametric marginal distributions to the data set and choose the one with the highest goodness-of-fit.

Step 2. Modeling dependencies over time

Copulas allow a fully parametric specification of the probability model. Specifically, the joint distribution function of a multivariate variable can be expressed as a multivariate function of univariate marginal distribution functions. For example, the joint distribution function of \(Y_i = (Y_{i1}, Y_{i2}, ..., Y_{iT})\) can be expressed as \(C(F_{i1}(y_{i1}), ..., F_{iT}(y_{iT}))\), where \(C\) is a copula and \(F_{i1}, ..., F_{iT}\) are marginal cumulative distributions.

Because of their tractability and ease in simulation, we employ two types of copulas: the \(t\)-copula, which is generated by the multivariate \(t\)-distribution, and the normal copula. By definition, the \(t\)-copula is parameterized by a correlation matrix \(\Sigma\) and degrees of freedom \(r\). Frees and Wang (2005) offer a fine framework for fitting copulas to multiple years of data, along with an overview of the \(t\)-copula and a discussion of various structures of the correlation matrix \(\Sigma\). In this paper three different forms of \(\Sigma\)
(the identity, the exchangeable, and the autoregressive with order 1–AR(1)) are fitted to the data. Maximum likelihood estimation is employed to estimate the parameters that are utilized in both the correlation matrix Σ and the parametric marginal distributions.

Step 3. Generating random draws using simulation

After fitting parametric copula models to historical loss data, one can employ predictive distributions to derive the conditional multivariate distribution of losses over multiple years into the future in order to determine the size and timing of the payout of the reinsurance contract. However, the complexity that is introduced by extending the loss distributions over multiple years makes the idea less practical than in the one-year case. Specifically, given a history of T years of data, the conditional distribution for the period \{T + 1,..., T + m\} is needed in order to analyze a multi-year contract covering m years. It is true that the t-copula possesses the property of tractability, i.e. the structure of the copula governing the relationships among the loss variables over the period being projected stays the same as that over the period of the historical data. However, the predictive density \(f(y_{i,T+1},...,y_{i,T+m}|y_{i,1},...,y_{i,T})\), and the resulting marginal distributions, are much more complex (see Appendix A.3 for the mathematical expression) and, therefore, much less convenient to apply to the modeling of the type of reinsurance contract we have described.

To address this issue, we utilize a simulation approach instead. After fitting marginal and joint distribution functions via copulas for both \((X_{i,1},...,X_{i,T})\) and \((Y_{i,1},...,Y_{i,T})\), we simulate the joint distribution of \((X_{i,T+1},...,X_{i,T+m})\) and \((Y_{i,T+1},...,Y_{i,T+m})\) separately using the marginal distributions and the copulas to be chosen in the process described in Section 2. The algorithm adopted can be found in Cherubini et al. (2004) and Embrechts et al. (2001). Further details are provided in Appendix B.

After a large number of simulations are generated, payouts from a multi-year, multi-line reinsurance contract can then be studied using Value at Risk and Conditional Tail Expectation measures.

3. DATASET OF COMMERCIAL MULTIPLE PERILS AND WORKERS COMPENSATION

To illustrate the proposed procedures, we make use of the loss experience derived from two lines of business, commercial multiple perils (CMP) and workers compensation (WC), reported by a number of property-casualty insurance companies in the U.S. The data source is Highline
Data (www.highlinedata.com) and it consists of statutory statements that insurance companies file with the National Association of Insurance Commissioners (NAIC). In the U.S, all insurance companies are required to file line of business loss experience reports regularly with the NAIC. According to Highline Data, a policy of commercial multiple perils refers to a contract that packages two or more insurance coverages designed to protect a commercial enterprise from various property and liability risk exposures. Workers compensation insurance covers an employer’s liability for injuries, disability, or death to employed persons, without regard to fault. We pick these two lines because they are less likely to be correlated (see the descriptive statistics presented in section 3.1), thus satisfying the across-line independence assumption, and enabling us to focus on temporal dependency. More details of the dataset are available in Appendix C.

Rather than dealing with loss amounts in dollars, we choose to model the loss ratios for each of these lines of business. Loss ratios are defined as the ratio of incurred losses to total premiums earned. In an effort to reduce the effects of individual firm characteristics on the loss experience, we selected companies with high credit ratings and significant market shares in both the CMP and the WC business lines. Future research can apply the same framework to actual loss amounts as the response variable and incorporate explanatory variables in order to better understand and model the loss behavior. We retain the notations CMP and WC for loss ratio variables and study five years of data for each insurer. The summary statistics of the loss ratio data are presented in Section 3.1. Section 3.2 examines the fitted marginal distributions, and Section 3.3 presents the fitted copulas.

In essence, the analysis is performed using the company as the unit of analysis. However, in practice, when applying this method, reinsurers and insurers should conduct the analysis using the contract as the unit of analysis, so that the result will be more reliable and relevant.

### 3.1 Descriptive statistics

Tables 1a and 1b display the descriptive statistics for both loss ratio variables, CMP and WC. The average loss ratio for commercial multiple perils varies from 46.965% in Year 5 to 68.956% in Year 2, and the range of the average loss ratio for workers compensation during the same period is 64.100% to 70.938%. One can also note that the maximum loss ratio for each line in all but one instance exceeds 100%.

The plots of both ratios versus Year are displayed in Figures 1a and 1b to shed light on any temporal trend. Due to the relatively short period of time under observation, an underwriting cycle is hardly evident. In the two plots, the line segments connect companies.
The Spearman correlations derived from the data set are presented in Tables 2a and 2b for CMP and WC, respectively. The loss ratios for the WC line display stronger correlations over time than CMP loss ratios but each line of business, taken separately, clearly demonstrates year-to-year dependency. In contrast, when the dependency between lines is studied, the Spearman correlation between CMP and WC turns out to be only 0.067, with the associated $p$-value of 0.40, confirming that cross-line interdependence is not a serious concern. We employ copulas to model the relationships over time.

### 3.2 Marginal distributions of loss ratios

The candidates for marginal distributions are normal, log-normal, and Gamma. Tables 3a and 3b display the goodness-of-fit statistics of each of

---

**Table 1a. Descriptive Statistics of CMP Loss Ratios (in percentage), by Year**

<table>
<thead>
<tr>
<th></th>
<th>CMP-Y1</th>
<th>CMP-Y2</th>
<th>CMP-Y3</th>
<th>CMP-Y4</th>
<th>CMP-Y5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>63.097</td>
<td>67.156</td>
<td>50.191</td>
<td>50.391</td>
<td>47.206</td>
</tr>
<tr>
<td>Median</td>
<td>60.800</td>
<td>64.000</td>
<td>50.800</td>
<td>50.400</td>
<td>42.250</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>13.375</td>
<td>15.683</td>
<td>11.319</td>
<td>11.252</td>
<td>15.277</td>
</tr>
<tr>
<td>Minimum</td>
<td>40.100</td>
<td>40.200</td>
<td>16.100</td>
<td>27.800</td>
<td>21.400</td>
</tr>
<tr>
<td>Maximum</td>
<td>104.200</td>
<td>103.100</td>
<td>67.800</td>
<td>76.800</td>
<td>108.200</td>
</tr>
</tbody>
</table>

**Table 1b. Descriptive Statistics of WC Loss Ratios (in percentage), by Year**

<table>
<thead>
<tr>
<th></th>
<th>WC-Y1</th>
<th>WC-Y2</th>
<th>WC-Y3</th>
<th>WC-Y4</th>
<th>WC-Y5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>63.016</td>
<td>70.675</td>
<td>67.091</td>
<td>70.672</td>
<td>68.297</td>
</tr>
<tr>
<td>Median</td>
<td>60.800</td>
<td>66.400</td>
<td>67.000</td>
<td>65.600</td>
<td>68.850</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>19.876</td>
<td>20.500</td>
<td>17.325</td>
<td>20.515</td>
<td>15.203</td>
</tr>
<tr>
<td>Minimum</td>
<td>24.900</td>
<td>32.400</td>
<td>30.400</td>
<td>44.400</td>
<td>38.900</td>
</tr>
<tr>
<td>Maximum</td>
<td>112.800</td>
<td>126.600</td>
<td>116.500</td>
<td>134.400</td>
<td>103.500</td>
</tr>
</tbody>
</table>
the candidate distributions for CMP and WC, respectively. Observation of the $p$-values indicates that both the log-normal and Gamma distributions provide reasonable fits to the loss ratios of CMP and WC, with the Gamma
fitting slightly better. To illustrate the procedures proposed, we use the Gamma distribution as the fitted marginal distributions for both loss ratio variables. The Gamma distribution has the flexibility to allow for long-tail distributions. This is confirmed in Figures 2a and 2b, showing the Q-Q plots of the two loss ratios, which demonstrate thicker tails at both ends of the fitted Gamma distributions.

### 3.3 Modeling dependence using copulas

We fit a $t$-copula along with a Gamma marginal distribution to both loss variables. Three different correlation matrices of the $t$-copula were fitted: identity, exchangeable, and AR(1). Since the time dimension consists of $T = 5$ years, the different structures of $\Sigma$ are actually:

| Table 2a. Spearman Correlations of Yearly CMP Loss Ratio |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                | CMP-Y1 | CMP-Y2 | CMP-Y3 | CMP-Y4 | CMP-Y5 |
| CMP-Y1        | 1.000  |        |        |        |        |
| CMP-Y2        | 0.421  | 1.000  |        |        |        |
| CMP-Y3        | 0.335  | 0.469  | 1.000  |        |        |
| CMP-Y4        | 0.090  | 0.191  | 0.485  | 1.000  |        |
| CMP-Y5        | 0.263  | 0.019  | 0.249  | 0.450  | 1.000  |

| Table 2b. Spearman Correlations of Yearly WC Loss Ratio |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                | WC-Y1 | WC-Y2 | WC-Y3 | WC-Y4 | WC-Y5 |
| WC-Y1         | 1.000  |        |        |        |        |
| WC-Y2         | 0.627  | 1.000  |        |        |        |
| WC-Y3         | 0.461  | 0.534  | 1.000  |        |        |
| WC-Y4         | 0.556  | 0.508  | 0.789  | 1.000  |        |
| WC-Y5         | 0.598  | 0.334  | 0.632  | 0.758  | 1.000  |
We employed maximum likelihood estimation to estimate all the parameters required—namely, the shape parameter $\alpha$ and the scale parameter $\gamma$ in the Gamma density function $f(x; \alpha, \gamma) = \frac{x^{\alpha-1}}{\gamma^\alpha \Gamma(\alpha)} \exp\left(-\frac{x}{\gamma}\right)$, as well as the degrees of freedom $r$ and the correlation coefficient $\rho$ associated with the $t$-copula in the case of the exchangeable and the autoregressive correlation matrices. Table 4 displays the values of the Akaike Information Criteria (AIC) to compare the goodness-of-fit. For this criterion, smaller values of AIC imply a better fit. For all three correlation matrices, AIC
Figure 2a. Q-Q plot of CMP loss ratio, Gamma distribution.

Figure 2b. Q-Q plot of WC loss ratio, Gamma distribution.
results from the $t$-copula are significantly less than the corresponding results from the normal copula, indicating that the $t$-copula provides a better fit than the normal model, regardless of the choice of the correlation matrix. This result is not surprising given that the $t$-copula has one free parameter more than the normal copula.

The estimated parameters of fitting the $t$-copula and the Gamma marginal distributions are displayed in Tables 5a and 5b. All parameters are statistically significant. Of major interest is the fact that all the correlation coefficients are statistically significant, providing strong evidence that the correlation structure in time is not independent. In the case of the WC line, the exchangeable correlation structure provides a better fit than both the AR(1) and the identity matrix. For the CMP line the AR(1) model outperforms the other two.

### 4. RISK MODELING BASED ON DIFFERENT TEMPORAL DEPENDENCIES

In this section, to estimate the CMP and WC loss ratios for each of three successive years, we simulate random draws from a multivariate distribution for three years using the fitted copula to represent the “year to year” dependence. For the purpose of comparison we also simulate the loss ratio variables assuming temporal independence. Then for a multi-year, multi-line reinsurance contract we analyze the distribution of the loss variable underlying the contract’s payout. The results clearly demonstrate that copulas effectively capture the effects of temporal dependence.
Table 5a. Maximum Likelihood Estimation Results for CMP by Correlation Matrix ($\Sigma$).

<table>
<thead>
<tr>
<th>Correlation matrix ($\Sigma$)</th>
<th>Identity</th>
<th>Exchangeable</th>
<th>AR(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correlation coefficient $\rho$</td>
<td>NA</td>
<td>0.147 (0.093)</td>
<td>0.434 (0.093)</td>
</tr>
<tr>
<td>Degrees of freedom $r$</td>
<td>4.226 (0.270)</td>
<td>4.232 (0.270)</td>
<td>4.252 (0.270)</td>
</tr>
<tr>
<td>Shape parameter $\alpha$</td>
<td>12.139 (1.570)</td>
<td>11.981 (1.594)</td>
<td>11.421 (1.613)</td>
</tr>
<tr>
<td>Scale parameter $\gamma$</td>
<td>4.632 (0.611)</td>
<td>4.722 (0.642)</td>
<td>4.981 (0.721)</td>
</tr>
<tr>
<td>AIC</td>
<td>995.92</td>
<td>994.38</td>
<td>979.07</td>
</tr>
</tbody>
</table>

Notes: Standard errors are reported in parentheses. The first two parameters are for the fitted $t$-copula; the other two are for the fitted Gamma marginal distribution.

Table 5b. Maximum Likelihood Estimation Results for WC by Correlation Matrix ($\Sigma$)

<table>
<thead>
<tr>
<th>Correlation matrix ($\Sigma$)</th>
<th>Identity</th>
<th>Exchangeable</th>
<th>AR(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>correlation coefficient $\rho$</td>
<td>NA</td>
<td>0.644 (0.091)</td>
<td>0.674 (0.075)</td>
</tr>
<tr>
<td>degrees of freedom $r$</td>
<td>4.164 (0.269)</td>
<td>4.236 (0.270)</td>
<td>4.255 (0.271)</td>
</tr>
<tr>
<td>shape parameter $\alpha$</td>
<td>13.999 (1.851)</td>
<td>10.655 (1.974)</td>
<td>10.922 (1.862)</td>
</tr>
<tr>
<td>scale parameter $\gamma$</td>
<td>5.038 (0.668)</td>
<td>6.644 (1.253)</td>
<td>6.425 (1.107)</td>
</tr>
<tr>
<td>AIC</td>
<td>1047.60</td>
<td>999.77</td>
<td>1003.42</td>
</tr>
</tbody>
</table>

Notes: Standard errors are reported in parentheses. The first two parameters are for the fitted $t$-copula; the other two are for the fitted Gamma marginal distribution.

4.1 Simulations using different temporal dependencies

We employ a multi-year multi-line reinsurance contract to illustrate the effects of temporal dependence and how copulas capture them. For the insurers in the dataset, the average CMP premium is about 25% greater
than the average WC premium, with both premiums increasing at an annual rate of 10%. We therefore have assumed that the primary insurer that is seeking reinsurance coverage expects to collect a first-year premium volume of $100 million from workers compensation insureds and $125 million from commercial multiple perils insureds. The primary insurer also expects both annual premium amounts will grow 10% each year for the next two years. It is assumed that the primary insurer is seeking a stop-loss reinsurance contract to cover aggregate losses arising from both CMP and WC over three years. The reinsurer of the contract will begin paying when the aggregate losses of the primary insurer from both lines over three years exceed a certain percentage, $d$, of the combined premiums earned by the primary insurer during the contract term. For example, when $d = 80\%$, the maximum aggregate loss of the primary insurer is capped at 80% of the total premiums collected from both coverages over the three years. Table 6 summaries the premium projections of the primary insurer.

Retaining the notations used in the previous sections, for the $t$-th year covered, $t = 1, 2, 3$, the payout of the reinsurance is equal to

$$\max \sum_{t=1}^{3} (P_{W,t} X_{T+t} + P_{C,t} Y_{T+t}) - D, 0$$

where $D$ is the stop-loss threshold, equal to total premium times the retention percentage. For instance, $D = 595.8$ when $d = 80\%$. The underlying variable that determines the contract payout is the overall loss the primary insurer incurs, ignoring the time value of money. That is, $Total \ loss = \sum_{t=1}^{3} (P_{W,t} X_{T+t} + P_{C,t} Y_{T+t})$.

The results of the analysis outlined in Section 3 indicate that the combination of a $t$-copula with an exchangeable correlation matrix and

### Table 6. Premium Projections (in $ million)

<table>
<thead>
<tr>
<th></th>
<th>1st year ($t = 1$)</th>
<th>2nd year ($t = 2$)</th>
<th>3rd year ($t = 3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>WC premium, $P_{W,t}$</td>
<td>100</td>
<td>110</td>
<td>121</td>
</tr>
<tr>
<td>CMP premium, $P_{C,t}$</td>
<td>125</td>
<td>137.5</td>
<td>151.25</td>
</tr>
<tr>
<td>Annual subtotal</td>
<td>225</td>
<td>247.5</td>
<td>272.25</td>
</tr>
<tr>
<td>Total premiums</td>
<td></td>
<td></td>
<td>744.75</td>
</tr>
</tbody>
</table>
marginal Gamma distribution provides a good fit for the multi-year WC loss ratios. For the CMP loss ratio, the best fit is achieved with the combination of a $t$-copula with an AR(1) correlation matrix and marginal Gamma distribution. We simulate random draws for the multivariate variables $(X_{T+1}, X_{T+2}, X_{T+3})$ and $(Y_{T+1}, Y_{T+2}, Y_{T+3})$, each represented by a $t$-copula, albeit with different correlation matrices.

Specifically, the WC loss ratios follow a marginal Gamma distribution with shape parameter $\alpha = 10.6546$ and scale parameter $\gamma = 6.6438$. The corresponding joint distribution over a three-year period is represented by a $t$-copula with degrees of freedom $r = 4.2362$ and a three-by-three exchangeable correlation matrix with $\rho = 0.6443$. For the CMP loss ratios, the Gamma marginal parameters were determined to be shape parameter $\alpha = 11.4205$ with scale parameter $\gamma = 4.9811$. The $t$-copula is parameterized by $r = 4.2524$ degrees of freedom and a three-by-three autoregressive correlation matrix with $\rho = 0.4339$.

We repeat the simulation procedure 10,000 times to generate potential results for each WC and CMP multivariate loss ratio separately. In this way, we are able to simulate the underlying loss variable of the reinsurance contract, Total loss, as defined above. For the purpose of comparison, we also generate simulations from multivariate loss ratios for each line separately using the fitted Gamma marginal distributions but assuming independence in time.

**4.2 Analysis of simulated Total loss variable**

Superimposed kernel densities of the Total loss variable generated under the two dependence structures are presented in Figure 3 to display the visual differences. The solid curve is the kernel density of Total loss using a copula dependence structure, while the dashed curve is the density assuming temporal independence. We note that the Total loss produced using copula dependence has thicker tails on both sides and is less symmetric than the distribution that results from assuming independence. Specifically, we note that the area to the right of 595.8, the stop-loss threshold, under the density curve of copula dependence is greater than the area under the density curve derived assuming temporal independence. Therefore, using a copula-based model will result in generating greater probabilities of the Total loss variable exceeding the stop-loss threshold.

In addition to this visual presentation, a Kolmogorov-Smirnov test and a Kuiper test were run to examine whether the Total loss variable generated by different methods would yield identical distributions. Neither test assumes any particular distribution for the variables being examined. Both tests produce $p$-values less than 0.0001, indicating the rejection of the null
hypothesis that assumes identical distributions will result. Parametric
distributions such as normal, log-normal, Gamma, and Weibull were
checked against the results of the Total loss simulations. None was a good
fit at the 1% level of significance when copula dependence was employed.
In the case of the independence assumption, Gamma was a good fit.

Descriptive statistics presented in Table 7 confirm what we have
observed in Figure 3. The distribution of Total Loss produced by an indepen-
dence assumption is more symmetric and less volatile compared to that
produced by assuming copula dependence. From the perspective of the
reinsurer, the magnitude of the right tail (“exceedance probability,” a key
term in catastrophe modeling) is the most significant issue. Table 7 shows
that the maximum value of Total loss under copula dependence is 12%
larger than that when temporal independence is assumed.
Although Value-at-Risk (VaR) is a measure commonly used to evaluate the potential loss of a high-risk asset or portfolio, it is not coherent in the terms defined by Artzner (1999). Recently, a robust, convenient, and coherent measure, conditional tail expectation (CTE), is quickly becoming a preferred measure for portfolio and asset assessment. CTE is especially useful in quantifying long-tail risk exposures. Manistre and Hancock (2005) summarize the properties of CTE that make it preferable to traditional Value-at-Risk measure. Tables 8a and 8b present the values of both VaR and CTE at high percentages to further study the differences in the distributions of the Total loss variable generated by assuming either copula dependence or temporal independence. To demonstrate the effects on risk measures of (i) an increasingly skewed distribution, and (ii) an increasingly dependent structure, two more sets of simulations were run and the associated VaRs and CTEs were calculated. Specifically, Table 8a displays the risk measures for three different temporal dependence structures: independence, dependence modeled by a $t$-copula with all parameters

### Table 7. Descriptive Statistics of Total Loss (in $ millions)

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Median</th>
<th>Standard deviation</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copula dependence</td>
<td>468.905</td>
<td>463.548</td>
<td>73.365</td>
<td>194.783</td>
<td>812.173</td>
</tr>
<tr>
<td>Independence</td>
<td>469.124</td>
<td>467.057</td>
<td>58.264</td>
<td>293.606</td>
<td>724.812</td>
</tr>
</tbody>
</table>

### Table 8a. VaR and CTE of Total Loss (in $ millions) Using Fitted Marginal Distributions, by Different Dependence Structures

<table>
<thead>
<tr>
<th>Percentage (%)</th>
<th>VaR</th>
<th>CTE</th>
<th>VaR</th>
<th>CTE</th>
<th>VaR</th>
<th>CTE</th>
</tr>
</thead>
<tbody>
<tr>
<td>99.5</td>
<td>698.080</td>
<td>732.394</td>
<td>683.025</td>
<td>719.132</td>
<td>631.948</td>
<td>655.245</td>
</tr>
<tr>
<td>99</td>
<td>660.840</td>
<td>704.613</td>
<td>649.589</td>
<td>692.477</td>
<td>610.872</td>
<td>637.249</td>
</tr>
<tr>
<td>95</td>
<td>595.420</td>
<td>637.094</td>
<td>587.857</td>
<td>627.161</td>
<td>568.998</td>
<td>595.911</td>
</tr>
<tr>
<td>90</td>
<td>563.536</td>
<td>607.559</td>
<td>557.461</td>
<td>599.084</td>
<td>545.428</td>
<td>576.016</td>
</tr>
</tbody>
</table>

Table 7. Descriptive Statistics of Total Loss (in $ millions)
being estimated ($\rho = .6443$ for WC and .4339 for CMP), and dependence modeled by a $t$-copula with arbitrarily reduced correlation coefficients ($\rho = 0.3$ for both WC and CMP). Table 8b illustrates the effect of introducing more skewed marginal distributions to the models. In this table, the fitted marginal Gamma distribution was modified by halving the shape parameter and doubling the scale parameter.

The observed values of both VaR and CTE support the notion that if temporal independence is assumed when analyzing the underlying loss variable of a multi-line, multi-year reinsurance contract, risk embedded in the product tends to be underestimated. As a result, decisions of asset-liability matching and capital management based on the risk assessment are likely to be inadequate, exposing the reinsurer to greater financial risk.

Consider the case where the stop-loss threshold is 80% of total premiums. Under this scenario, the reinsurance coverage will be triggered when the aggregate loss of the primary insurer exceeds $595.8 million. Of 10,000 simulations of Total loss based on the assumption of temporal independence, 196 are greater than the threshold, indicating that the reinsurer should expect claims from the primary insurer at a frequency of approximately 2%, or one every fifty years. The aggregate payments made by the reinsurer as a result of exceeding the threshold 196 times is $4,805 million, leading to an average expected claim of $24.5 million.

However, when 10,000 simulations were run assuming the fitted copula dependence ($\rho = .64$ for WC and .43 for CMP), 495 exceeded the threshold, yielding an expected reinsurer claim frequency of approximately 5%, or one every twenty years. The average amount of expected
claims is $41.71 million, 70% higher than the estimation based on temporal independence assumption. When the correlation coefficients in the copula correlation matrix are reduced, the CTEs and VaRs are still significantly larger than those derived under the independence assumption, though only slightly smaller than the case of the fitted dependence structure.

Comparing the results in both tables, we note that an increasingly skewed marginal distribution (Table 8b) also has significant influences on the risk measures, even after controlling for the dependence structure. For example, when the dependence structure is the fitted copula model, the CTE calculated using the fitted marginal Gamma at 99.5% significance level is $732.394 million. When the marginal Gamma’s skewness coefficient is increased to 1.414 times the fitted value, the CTE jumps to $861.940 million.

5. SUMMARY AND CONCLUDING REMARKS

The relationships between loss variables underlying a multi-year, multi-line reinsurance contract are complicated since correlations may be present in two dimensions: from year to year and/or across different business lines. Thus, modeling of the underlying loss variables of the reinsurance contract presents a challenge. While the reinsurer of the multi-year, multi-line agreement may be able to choose two or more independent business lines for the purpose of risk diversification, it is nearly impossible to remove all of the temporal dependence of annual loss experience. The assumption of temporal independence is likely to result in a misidentification of the distribution of the underlying loss variables, potentially leading to a significant under-estimation of the risk embedded in the reinsurance product.

This paper applies copulas to the modeling of the multi-year dependencies of losses arising from each business line covered by the reinsurance contract, simulates random draws from multivariate loss ratios represented by the copulas, and demonstrates the effects of temporal dependencies on some risk measures of a multi-year, multi-line reinsurance contract.

As a matter of fact, the same approach can be readily extended to address simultaneously both temporal dependencies and dependencies among lines of business. Specifically, the correlation matrix (\( \Sigma \)) in Section 3.3 can be extended to a 10 by 10 matrix in which the two sub-matrices on the diagonal capture the temporal dependencies for CMP and WC respectively (as illustrated), while the two sub-matrices off the diagonal model

---

2 This is achieved by halving the shape parameter and doubling the scale parameter so that the mean value remains unchanged.
the dependencies among the two lines of business. See, for example, Shi and Zhang (2011) for a similar application.

The idea proposed provides an improved and convenient tool for modeling the risk of alternative risk transfer products. If individual insurer characteristics associated with the loss ratio variable can be identified, the analysis can be extended to the application that incorporates covariates in order to better predict that insurer’s loss distribution.

ACKNOWLEDGEMENTS

The Society of Actuaries (through the Committee on Knowledge Extension Research) and the Actuarial Foundation (through the Actuarial Education and Research Fund) provided funding to support this research. The author is grateful for the support.

REFERENCES


APPENDIX A

Predictive Distribution Using t-Copula

Frees and Wang (2005) present a brief review of the t-copula and its related predictive distribution. In this appendix, the predictive distribution is extended to multivariate case for the completeness of the paper.

A.1 The multivariate t-distribution

Suppose that \( (N_1, \ldots, N_T)' \) has a joint standardized multivariate normal distribution with correlation matrix \( \Sigma \). Also assume that \( \chi_r^2 \) follows a Chi-square distribution with \( r \) degrees of freedom and is independent of \( (N_1, \ldots, N_T)' \). Then, the joint distribution of \( \{Z_t = N_t(\chi_r^2/\sqrt{r})^{-1}, t = 1, \ldots, T\} \) constitutes a multivariate t-distribution with \( r \) degrees of freedom. One property of this distribution is that each marginal distribution is a t-distribution with \( r \) degrees of freedom, denoted by \( T_r \). Moreover, subsets have the same family as the joint distribution. Thus, if we assume that \( (Z_{1t}, \ldots, Z_{Tt}) \) follows a multivariate t-distribution, then \( (Z_{1t}, \ldots, Z_{Tt}) \) also has a multivariate t-distribution. The joint probability density function of \( (Z_{1t}, \ldots, Z_{Tt})' \) is

\[
f_z(z; r, \Sigma) = \frac{\Gamma((r + T)/2))}{(r\pi)^{T/2}\Gamma(r/2)|\Sigma|^{1/2}}\left(1 + \frac{1}{r} z'\Sigma^{-1} z\right)^{-\frac{r + T}{2}}
\]

where \( z = (z_{1t}, \ldots, z_{Tt})' \).

A.2 The t-copula

Multivariate t-copula is a function defined for all \( (u_1, u_2, \ldots, u_T) \in [0,1]^T \) by \( C(u_1, \ldots, u_T) = F_z(T_r^{-1}(u_1), \ldots, T_r^{-1}(u_T)) \), where \( F_z \) is the cumulative distribution function associated with the probability density function \( f_z \) of multivariate t-distribution. From equation (A.1), the corresponding probability density function is

\[
c(u_1, \ldots, u_T) = f_z(T_r^{-1}(u_1), \ldots, T_r^{-1}(u_T)) \prod_{t=1}^{T} \frac{1}{t_r(T_r^{-1}(u_t))}, \text{where } t_r(.) \text{ is the probability density function associated with } T_r, \text{ a univariate t-distribution with } r \text{ degrees of freedom. The conditional density function using copula is}
\]
A.3 Predictive density

If we assume the distribution function of \( Y_i = (Y_{i1}, Y_{i2}, Y_{i3}, \ldots, Y_{iT})' \) can be modeled by a copula, we have \( F_i(y_{i1}, \ldots, y_{iT}) = C(F_{i1}(y_{i1}), \ldots, F_{iT}(y_{iT})) \). The related joint density function is given by

\[
 f_i(y_{i1}, \ldots, y_{iT}) = c(F_{i1}, F_{i2}, \ldots, F_{iT}) \prod_{t=1}^{T} f(y_{it}, \theta_{it}), \text{ where } \\
 f_{it}(y_{it}) = f(y_{it}, \theta_{it}) \text{ and } F_{it}(y_{it}) = F_{it} \text{ is the corresponding distribution function. Thus, the predictive distribution for the period } \{ T+1, \ldots, T+m \} \text{ is }
\]

\[
 f_i(y_{i,T+1}, \ldots, y_{i,T+m}|y_{i1}, \ldots, y_{iT}) = \frac{f_i(y_{i1}, \ldots, y_{iT}, y_{i,T+m})}{f_i(y_{i1}, \ldots, y_{iT})} \\
= \frac{c(F_{i1}, \ldots, F_{iT}, \ldots, F_{i(T+m)})}{c(F_{i1}, \ldots, F_{iT})} \prod_{t=1}^{m} f(y_{i,T+t}, \theta_{i,T+t}) \\
= f_z(v_{i,T+1}, \ldots, v_{i,T+m}|v_{i1}, \ldots, v_{iT}) \prod_{s=1}^{m} \prod_{t=1}^{m} f(y_{i,T+s}, \theta_{i,T+s}) \\
\]

where \( v_{it} = T_r^{-1}(F_{it}(y_{it})) \), \( t = 1, \ldots, T+m \) and \( v_i = (v_{i1}, \ldots, v_{iT})' \).

APPENDIX B

Simulation of multivariate variable using t-copula

For a 3-dimensional multivariate whose marginal distributions are \( F_1, F_2, F_3 \) and joint distribution is represented by a t-copula with \( r \) degrees of freedom and correlation matrix \( \Sigma \), the simulation procedure follows.

- Find the Cholesky decomposition \( A \) of \( \Sigma \)
- Simulate three \( i.i.d \) \( z_1, z_2, z_3 \) from standard normal \( N(0,1) \) to form \( z = (z_1, z_2, z_3)' \)
• Simulate a variate $s$ from $\chi^2_r$ independent of $z$, where $\chi^2_r$ is Chi-square distribution with $r$ degrees of freedom

• Set $y = Az$

• Set $u = \sqrt{(r/s)}y$

• Set $t_i = T_r(u_i)$ for $i = 1, 2, 3$, where $T_r$ denotes the cumulative distribution function of univariate $t$-distribution with $r$ degrees of freedom

• $(x_1, x_2, x_3)' = (F_1^{-1}(t_1), F_2^{-1}(t_2), F_3^{-1}(t_3))'$ represents a random draw.

APPENDIX C

Companies in the Data Set

- Cincinnati Insurance Co.
- Tokio Marine & Nichido Fire Ins Co
- Federated Mutual Insurance Co
- Frankenmuth Mutual Insurance Co
- Harleysville Mutual Insurance Co
- Hastings Mutual Insurance Co
- GuideOne Mutual Insurance Co
- Public Service Mutual Insurance Co
- Society Insurance
- Church Mutual Insurance Co
- Auto Owners Insurance Co
- American Family Mutual Insurance Co
- Hartford Fire Insurance Co
- Central Mutual Insurance Co
- Federal Insurance Co
- Pacific Indemnity Co
- Vigilant Insurance Co
- Continental Casualty Co
- Country Mutual Insurance Co
- Farmers Insurance Exchange
- Hanover Insurance Co
- Secura Insurance A Mutual Co
- New Hampshire Insurance Co
- Ohio Casualty Insurance Co
- Westfield Insurance Co
- Pekin Insurance Co
- General Casualty Co of Wisconsin
- State Farm Fire and Casualty Co
- Charter Oak Fire Insurance Co
- Utica Mutual Insurance Co
- Erie Insurance Exchange
- Hartford Casualty Insurance Co

The market share of each company in the list in both lines of business (commercial multiple perils and workers compensation) was in the top 200 in year 2006. The time period observed is 2000–2004, corresponding to Year 1 through Year 5 in the analysis.