Optimal Insurance Pricing, Reinsurance, and Investment for a Jump Diffusion Risk Process under a Competitive Market

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Abstract: We extend previous research by considering the role of reinsurance in hedging underwriting risk, pricing risk, and investment risk. We consider a stochastic dynamic optimization model applied to the problem of insurance pricing under a competitive insurance market with a jump diffusion risk process. Our model seeks to maximize the expected utility of the insurer’s terminal wealth, incorporating the interaction of a stochastic process for the insurance price evolution, reinsurance, investment strategy, and the possible hedging effect between insurance liabilities and investment risk. We solve this optimization problem by constructing a Hamilton–Jacobi–Bellman (HJB) equation. [Key words: stochastic dynamic optimization, insurance demand, insurance price, investment portfolio, jump diffusion.]

INTRODUCTION

Optimal pricing of insurance contracts, often in conjunction with other insurer operations including reinsurance and investment, has been examined widely in the literature. For example, prior work has solved for the insurer’s optimal investment portfolio or the optimal investment-

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reinsurance strategy under the assumption that the price of insurance (per unit of exposure) is given (see Taylor, 1986; Emms, 2007a; Emms, Haberman, and Savoulli, 2007; and Emms and Haberman, 2005). In reality, however, the risks and returns of insurer investment portfolios vary and may affect the price of insurance. Additionally, the quantity of insurance demanded will affect the insurance price, and, under some circumstances, the risk of the liabilities assumed by the insurer may be balanced with the insurer’s investment risk.

Other studies that have considered the insurer’s optimal investment strategy, the insurer’s joint investment and reinsurance strategy, and the hedging value of reinsurance include Browne (1995), Hipp and Plum (2000), Liu and Yang (2004), Browne (2000), Korn (2005), Korn and Seifried (2009), Mataramvura and Åksendal (2008), Promislow and Young (2005), Zhang and Siu (2009), Cao and Wan (2009), and Scordis and Steinhorth (2012). Schmidli (2002) considered a classical model with drift and discusses the optimal decision on investment and reinsurance strategy by establishing the objective function of minimizing the ruin probability via solving a Hamilton–Jacobi–Bellman (HJB) equation. Luo, Taksar, and Tsoi (2008) considered a problem of optimal reinsurance and investment for an insurer whose surplus is governed by a linear diffusion. Their main goal is to find the optimal investment and reinsurance strategy that minimizes the probability of ruin.

Edoli and Runggaldier (2010) studied the optimization of reinsurance and investment. They assume that the arrival of a claim and the change of the price of the underlying asset(s) corresponds to a Poisson point process. The objective is to maximize the expected total utility. And as a special case, they also discuss maximizing exponential utility functions whereby negative values of the risk process are penalized. Eisenberg and Schmidli (2011) studied optimal control of a classical risk model and its diffusion approximation. They assume that the individual claims are reinsured fully or partially and they also assume the insurer is allowed to invest in a riskless asset with some constant interest rate. The objective is to minimize the discounted capital injections. They found explicit optimal solutions by solving an HJB equation. Liu, Yiu, Siu, and Ching (2013) studied an optimal investment-reinsurance problem for an insurer who faces dynamic risk constraint in a Markovian regime-switching environment.

Lin and Li (2011) considered an optimal reinsurance-investment problem of an insurer whose surplus process follows a jump-diffusion model. The dynamics of the risky asset are governed by a constant elasticity of variance model to incorporate conditional heteroscedasticity. The objective of the insurer is to choose an optimal reinsurance-investment strategy so as to maximize the expected exponential utility of terminal wealth. Their
study does not consider the effect of the insurer’s investment strategy on the price of insurance, nor the effect of the insurance price as a stochastic process on the investment strategy, and the possible hedging effect, which was represented by increasing the number of policies written to hedge investment risk.

Mao et al. (2013) incorporated investment risk and the market average price uncertainty in the pricing decision of an insurer. They assumed that the price, the investment, and the insured losses are stochastic processes, while simultaneously considering the effect of demand on price, assuming constant price elasticity of demand. However, they did not consider the effect of reinsurance on pricing and investment strategy and on the expected utility of terminal wealth of an insurer, nor their roles as potential cross-hedges of risks faced by the insurer. The study also did not consider the effect of the shock of big claims risk on the expected utility of terminal wealth of the insurer.

In this paper, we extend Carson et al. (2016) and other literature by considering the role of reinsurance in hedging underwriting risk, pricing risk, and investment risk. We also consider the inter-hedging of reinsurance, pricing, and investment risk. For simplicity, we assume that all insurers sell insurance based on the same price process, and this actually results in a perfectly competitive market, with all insurers charging the same price. For the claim process, we follow the assumption used by Lin and Li (2011); that is, we assume that the claim process is a jump diffusion process that considers extreme claim events. We construct a Hamilton–Jacobi–Bellman equation and solve it to determine the optimal price of insurance, the optimal investment portfolio, and the insurer’s optimal reinsurance strategy.

Insurers face numerous risks, such as pricing risk, investment risk, and underwriting risk. Those risks generally are correlated and change dynamically. Insurer management is increasingly held responsible for managing those risks in an integrated and systematic fashion due to competitive pressures, expectations of the company’s owners, and regulatory requirements. One particular such regulatory requirement, emerging now globally, is the Own Risk and Solvency Assessment (ORSA). In the United States, under the Risk Management and Own Risk and Solvency Assessment Model Act, starting in 2015 large and medium-size U.S. insurers and insurance groups are required to regularly perform an ORSA and file a

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4A commonly used assumption in economic literature is that of a linear demand function. Of course this assumption is restrictive and represents simplification of the complexity of economic interaction of buyers and sellers, but it has the advantage of being intuitively appealing and giving a range of elasticities of demand.
confidential ORSA Summary Report with insurer regulators upon request, and with the lead state regulator for each insurance group whether or not any request is made (see NAIC, 2016). A similar integrated risk management process is required under the Solvency II insurance regulation in the European Union that became effective in 2016.

This paper is organized as follows: The next section presents the models. Then in the following section the HJB equation is given and the optimal price, investment portfolio, and reinsurance strategy are determined. We then perform the sensitivity analysis in the next section, and in the final section we provide the summary and conclusions.

THE MODELS

In a manner analogous to that of Mao et al. (2013), we assume that the insurer can invest its wealth in one risk-free asset and one risky asset, which can be traded continuously over time, without any transaction costs or taxes. Let \{ \( W_1(t); t \in [0,T] \) \} be a standard Brownian motion on a filtered probability space. We then assume that the price process of risky investment, i.e., \( S(t) \), evolves over time according to the following constant variance model:

\[
dS(t) = \mu S(t) dt + \sigma S(t) dW_1(t), \quad \text{with} \quad S(0) = s_0,
\]

where \( \mu \) is the drift of the process and \( \sigma \) is the volatility of the process. The price of the risk-free asset is assumed to evolve according to the equation

\[
 dB(t) = rB(t) dt,
\]

where \( r \) is the risk-free interest rate, with \( r < \mu \). The average market price of insurance \( \bar{p} \) (per unit of exposure) is assumed to follow a geometric Brownian motion satisfying

\[
d\bar{p}(t) = \bar{p}(t)(\mu_p dt + \sigma_p dW_3(t)),
\]

where \( \mu_p \) and \( \sigma_p \) are appropriate drift and volatility parameters. We also assume that the price \( p \) for insurance charged by a specific company is determined from the value of \( \bar{p} \) via

\[
p(t) = kp(t)
\]
with \( k \) being a company-specific fixed parameter.

We also assume that the insurer’s surplus without reinsurance and investment at time \( t \) follows a jump diffusion stochastic process as:

\[
R(t) = x + p(t) - \sum_{i=1}^{N(t)} Y_i + \theta W_2(t)
\]  

(5)

where the constant \( x \) is the initial surplus; \( p(t) \) is the price of insurance product; \( \{N(t): t \in [0,T]\} \) is a Poisson process with constant intensity \( \lambda \), where \( \lambda > 0 \) and \( N(t) \) represents the number of claims up to time \( t \); \( \{Y_i, i = 1, 2, \ldots\} \) is a sequence of independent and identically distributed nonnegative random variables with a common distribution \( G(y) \) having a finite mean \( v \), where \( Y_i \) is the amount of the \( i \)-th claim; \( \{W_1(t); t \in [0,T]\} \) is a standard Brownian motion on a filtered probability space; the constant \( \theta \) represents the volatility parameter (or the diffusion parameter).

The price of the insurance product affects the quantity demanded, and the relationship is expressed by the demand function. Mao et al. (2013) discuss the dynamic pricing of insurance under the assumption of constant price elasticity of demand. The linear demand functions are also often used in economics literature (e.g., Lilien and Kotler, 1983; Mankiw, 2008), and we use it here in the form

\[
p(t) = D - Eq(t),
\]

(6)

where \( D \) and \( E \) are constants, \( q(t) \) is the demand for insurance product at time \( t \), and \( p(t) \) satisfies its stochastic differential equation:

\[
dp(t) = kp(t)(\mu_p dt + \sigma_p dW_3(t)) = p(t)(\mu_p dt + \sigma_p dW_3(t)).
\]

We also assume that insurance contracts are independent of each other.

Let \( \{X(t): t \in [0,T]\} \) be the insurer’s wealth process. The wealth process of the insurer who sells the quantity of insurance contracts, \( q(t) \), at time \( t \) and reinsures the proportion \( (1-a(t)) \) to the reinsurer can be described by the following stochastic differential equations:
\[ dX(t) = (p(t)q(t)a(t)(1 + \eta) - \eta) + \pi(t)(\mu - r) + X(t)r dt + \]
\[ \sigma \pi(t)dW_1(t) + \theta dW_2(t) - d\left( q(t) \times a(t) \times \sum_{i=1}^{N(t)} Y_i \right) d\tilde{p}(t) = \]
\[ \tilde{p}(t)(\mu_p dt + \sigma_p dW_3(t)) \]

with \( p(t) = k(t)\tilde{p}(t) \) and \( X(t_0) = x \), where \{ \( W_1(t), W_3(t) | F_t, t \geq 0 \} \) are two correlated and standard Brownian motions on a filtered probability space, and \( F_t \) is the \( P \)-augmentation of the natural filtration, \( a(t) \) is the proportion of retention at time \( t \), and \( \eta \) is the gross premium rate of reinsurance.

Furthermore, the correlation matrix is
\[
\begin{pmatrix}
1 & \rho_{13} \\
\rho_{31} & 1
\end{pmatrix}
\]
\( W_2(t) \) is an independent stochastic process and \( \pi(t) \) is the allocation of the investment portfolio to the risky investment at time \( t \). Similar to Mao et al. (2013), we assume that \( \pi(t) \) may be less than zero, i.e., short selling is permitted.\(^5\) We also assume that borrowing money is allowed, i.e., \( \pi(t) \) can be larger than \( -X(t) \), with the risk-free rate being the borrowing cost.

Equation (7) describes the dynamic wealth process of an insurer. The first (right-hand-side) term is the sum of retained premium income minus the cost of reinsurance plus the return on risky and risk-free investments. The second term expresses the volatility of risky investment and the last term in Equation (7) is a jump process of claim loss.

**HJB EQUATION AND ITS SOLUTIONS FOR OPTIMAL PRICE, OPTIMAL INVESTMENT, AND REINSURANCE STRATEGY**

We study the optimal pricing of insurance and the insurer’s optimal investment and reinsurance strategy under the assumption that the quantity demanded is a linear function of price. We formulate the problem of maximizing the expected utility of the insurer’s terminal wealth, given

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\(^5\)Short selling is an investment strategy equivalent to assuming a negative position in an asset. Besides Mao et al. (2013), several other papers also make such an assumption (e.g., Browne, 1995; Promislow and Young, 2005; and Zhang and Siu, 2009).
the initial values of time, $t_0$, and the initial wealth of the insurer, $X_0$. The objective function over the class of admissible controls $A_{t_0,X_0,p_0}$ is given by

$$J((t_0 = 0, X_0 = x, p_0 = p_0); (\pi,p)) = E_{t_0 = 0, X_0 = x, p_0 = p_0}(U(X(T))).$$

(8)

The optimization problem is to find the value function $V(t,X,p)$ and optimal solutions $(\pi,p) \in A_{t_0,X_0,p_0}$ that satisfy the condition

$$V(t,X,p) = \sup_{(\pi,a,p) \in A_{t,X,p}} (J(t,X,p);(\pi,p)).$$

(9)

It can be shown easily that $V(t,X,p)$ is a Markov process. For any twice continuously differentiable function $h \in C(1,2)(O) \cap C(\overline{O})$, where $O = (0,T) \times (0,\infty) \times (0,\infty)$ and $\overline{O}$ denotes the closure of $O$, there exists a partial differential operator $L^{\pi,a,p}(h(t,x,p))$:

$$L^{\pi,a,p}(h(t,x,p)) = \frac{\partial h}{\partial t} + \left(\frac{p(D-p)}{E}\right)(a(1+\eta) - \eta) + (\mu - r)\pi + rx \frac{\partial h}{\partial x} +$$

$$+ \frac{1}{2} \sigma^2 \sigma^2 + \theta^2 \frac{\partial^2 h}{\partial x^2} + \left(\rho_{13} \pi \sigma \sigma \frac{D-p}{E}\right) \frac{\partial^2 h}{\partial \pi \partial \sigma} +$$

$$+ \frac{1}{2} \rho_{23}^2 \rho_{23}^2 \frac{\partial^2 h}{\partial \sigma^2} + \lambda \frac{D-p}{E} \int_0^\infty V(t,p,x - ay, \frac{D-p}{E}) -$$

$$V(t,p,x) G(dy \cdot \frac{D-p}{E}).$$

The following verification theorem can be obtained.

Theorem 1: Suppose there exists a function $\phi_D(t,x,p) \in C(1,2)(O) \cap C(\overline{O})$ and a control (i.e., an optimal solution) $(\pi^*, p^*) \in A$ such that

1. $L^{\pi,a,p}(\phi(t,x,p)) \geq 0$ for all $(\pi,p) \in A$ and $(t,x) \in O$;
2. $L^{\pi,a,p}(\phi(t,x,p)) = 0$ for all $(t,x) \in O$;
3. For all $(\pi,p) \in A$, $\lim_{t \to T} \phi(t,x,p) = U(X^{\pi,p}(T))$;
4. Let $\kappa$ denote the set of stopping times $\tau < T$. The family $\{\phi(\tau,x,p)\}_{\tau \in \kappa}$ is uniformly integrable for all $(\tau,x,p) \in O$, $(\pi,p) \in A$. Then $\phi(\tau,x,p) = V(t,x,p)$, and $(\pi^*, a^*, p^*)$ is an optimal control.

Now, let us formulate the optimal pricing, investment, and reinsurance problem of maximizing the expected utility of the insurer’s terminal
wealth. We consider the case of exponential utility, i.e., we assume that the insurer's utility function is an exponential utility function defined by:

$$U(x) = \frac{e^{-\gamma x}}{\gamma},$$  \hspace{1cm} (11)

where $\gamma$ is a positive constant and represents the coefficient of absolute risk aversion. This assumption, similar to our linear demand assumption, is also somewhat restrictive, but it is reasonable and has been used in prior literature (e.g., Mao et al., 2013). In order to obtain the optimal value function $V(t,x,p)$ and the optimal strategy $(\pi^*,a^*,p^*)$, we need to solve the following HJB equation (see, e.g., Fleming and Rishel (1975)):

$$\begin{cases}
\sup_{(\pi,a,p) \in A} L^{\pi,a,p}(V(t,x,p)) = 0, \\
V(t,x,p) = -\frac{e^{-\gamma(x + pq(a(1 + \eta) - \eta))}}{\gamma}.
\end{cases}$$  \hspace{1cm} (12)

To solve the above HJB equation, we try a solution of the following form:

$$\phi^1(t,x,p) = -\exp(-\gamma(x + pq(a(1 + \eta) - \eta)) \frac{e^{r(T-t)} + f(t))}{\gamma},$$  \hspace{1cm} (13)

where $f(t)$ is an undetermined function with $f(T) = 0$. We take partial derivatives of the first and second order of equation (13) with respect to $x$, $t$, and $p$, and get the following:

$$\frac{\partial \phi^1}{\partial x} = -\phi^1 \gamma e^{r(T-t)},$$  \hspace{1cm} (14)

$$\frac{\partial \phi^1}{\partial t} = \frac{\partial \phi^1}{\partial t}(r(x + pq)\gamma e^{r(T-t)} - \mu_p \tilde{p}(q - k\tilde{p})\gamma e^{r(T-t)} + f_t),$$

$$\frac{\partial \phi^1}{\partial \tilde{p}} = -\phi^1 \gamma e^{r(T-t)}(\frac{D - 2k\tilde{p}}{E})(a(1 + \eta) - \eta),$$

$$\frac{\partial \phi^1}{\partial x^2} = \phi^1 (\gamma e^{r(T-t)})^2,$$

$$\frac{\partial \phi^1}{\partial x \partial p} = \phi^1 (\gamma e^{r(T-t)})^2 (\frac{D - 2k\tilde{p}}{E})(a(1 + \eta) - \eta),$$

$$\frac{\partial \phi^1}{\partial p^2} = \gamma e^{r(T-t)}((\frac{D - 2k\tilde{p}}{E})^2 (a(1 + \eta) - \eta)\gamma e^{r(T-t)} + \frac{2k}{E})(a(1 + \eta) - \eta)$$

$$V(t,p,x - \eta y) - V(t,p,x) = (e^{\gamma\eta y e^{r(T-t)}} - 1)V.$$
Putting the above trial function, equation (4), and equation (14) into equation (10) yields:

\[
L^\pi.\hat{a}.p\left(\phi^1(t, x, \hat{p})\right) = \phi^1(t, x, \hat{p}). \tag{15}
\]

\[
\left(f_1 - \gamma e^{r(T-t)}\left(\frac{p(D-p)}{E}\right)(a(1 + \eta) - \eta) + (\mu - r)\pi - \right.
\]

\[
\left(\frac{rp}{k} \cdot \frac{D-p}{E} - \frac{\pi p p}{E} \left(\frac{D-2p}{E}\right)\right)(a(1 + \eta) - \eta) + \right.
\]

\[
\left(\frac{1}{2}(\sigma^2 + \theta^2)\gamma e^{r(T-t)}\right)^2 + \frac{1}{2}(\sigma^2 + \theta^2)\gamma e^{r(T-t)} + \right.
\]

\[
\left(\rho_{13}\pi \left[\frac{D-p}{E} \sigma \sigma_p\right]\gamma e^{r(T-t)}\right)^2 \cdot \frac{D-2p}{E} (a(1 + \eta) - \eta) - \right.
\]

\[
\mu_{p p} \cdot \frac{D-p}{E} \cdot \frac{D-2p}{E} (a(1 + \eta) - \eta) \cdot \gamma e^{r(T-t)} + \right.
\]

\[
\left(\frac{1}{2}\sigma^2 \gamma e^{r(T-t)}\left(\frac{D-2p}{E}\right)^2 (a(1 + \eta) - \eta) (\gamma e^{r(T-t)}) + \frac{2k}{E}\right)
\]

\[
(a(1 + \eta) - \eta) + \lambda \frac{D-p}{E} \int_0^{\infty} \left(e^{\gamma a e^{r(T-t)}} - 1\right) G(dy)
\]

We simplify the analysis and assume that the insurance market is competitive and that all insurance firms charge the same price, that is,

\[
\sum_{i=1}^{N} p_i(t) = \frac{1}{N} \sum_{i=1}^{N} k p_i(t) = \frac{1}{N} \sum_{i=1}^{N} k p_i(t) = \hat{p}(t), \text{ implying that } k = 1. \text{ The insurance price } p \text{ in our model is assumed to equal the average price, } \hat{p}. \text{ From that, we can assume that } \frac{2k}{E} \text{ in equation (15) is equal to } \frac{2}{E}. \text{ This allows us to set}
\]

\[
\frac{rp}{k} \cdot \frac{D-p}{E} = rp \cdot \frac{D-p}{E} \text{ in order to simplify the calculations. The first order conditions for optimal control } (\pi^*(t), a^*(t), p^*(t)) \text{ are:}
\[
\left\{ \frac{D-2p}{E}(a(1+\eta) - \eta) + \frac{p(D-p)(1+\eta)}{E} \frac{\partial a}{\partial p} \right. \\
+ (\mu - r) \frac{\partial \pi}{\partial p} \left( r p \cdot \frac{D-p}{E} - \mu_p \left( \frac{D-2p}{E} \right) \right) (1+\eta) \frac{\partial a}{\partial p} - \left( \frac{D-2p}{E}(1-r) + \mu_p \cdot \frac{D-3p}{E} \right) (a(1+\eta) - \eta) \\
\left. + \frac{1}{2} \gamma e^{r(T-t)} \left( 2\sigma^2 \frac{\partial \pi}{\partial p} \right) \right\} \\
\gamma e^{r(T-t)} \left( \rho_{13} \sigma p \left[ \frac{\partial \pi}{\partial p} + \sqrt{\frac{D-p}{E} - \pi} \frac{1}{2\sqrt{\frac{D-p}{E}}} \right] p \cdot \frac{D-2p}{E} \right) (a(1+\eta) - \eta) + \\
\left. + \rho_{13} \pi \sigma p, \left[ \frac{D-p}{E} \cdot \frac{D-2p}{E} \right] (1+\eta) \cdot \frac{\partial \pi}{\partial p} \right) + \\
\mu_p \left( \frac{2p(D-p)}{E^2} - \left( \frac{D-2p}{E} \right)^2 \right) (a(1+\eta) - \eta) + \mu_p p \cdot \frac{D-p}{E} \frac{D-2p}{E} (1+\eta) \frac{\partial a}{\partial p} + \\
\frac{1}{2} \frac{2p^2}{\sigma^2} \left( \frac{p(2D-3p)}{E} \left( \left( \frac{D-2p}{E} \right)^2 (a(1+\eta) - \eta) \gamma e^{r(T-t)} + \frac{2}{E} \right) \right) (a(1+\eta) - \eta) + \\
- \frac{4p^2}{\sigma^2} \left( \frac{D-p}{E} \frac{D-2p}{E} \right) (a(1+\eta) - \eta) \gamma e^{r(T-t)} + \frac{1}{E} \right) (1+\eta) \frac{\partial a}{\partial p} + \\
\frac{\lambda}{ \gamma e^{r(T-t)}} \left( \frac{\partial a}{\partial p} \cdot \frac{D-p}{E} \gamma e^{r(T-t)} - \frac{1}{E} \right) \int_0^{\infty} ye^{\gamma e^{r(T-t)}} G(dy) = 0
\]

where

\[
\pi(t) = \frac{1}{\sigma} \left( \frac{\mu - r}{\gamma e^{r(T-t)}} - \rho_{13} \left[ \frac{D-p}{E} p \sigma \cdot \frac{D-2p}{E} (a(1+\eta) - \eta) \right] \right) \tag{17}
\]

\[
\frac{\partial \pi}{\partial p} = \frac{1}{\sigma \rho_{13} \sigma p} \left\{ p \cdot \frac{D-2}{E} \sqrt{\frac{D-p}{E}} + \left[ \frac{D-p}{E} \cdot \frac{D-2p}{E} \right] \left( \frac{D-p}{E} \right)^2 \right\} \tag{18}
\]

\[
(a(1+\eta) - \eta) - \frac{1}{\sigma} \left[ \rho_{13} \left[ \frac{D-p}{E} p \sigma \cdot \frac{D-2p}{E} (1+\eta) \frac{\partial a}{\partial p} \right] \right]
\]
\begin{equation}
 a(t) = \frac{C\left(\eta\left(\frac{D-2p}{E}\right)^2 - 1/E\right) + B - A - D_1}{(1 + \eta)C\left(\frac{D-2p}{E}\right)^2},
 \end{equation}

where

\begin{align*}
 A &= \left(\rho_{13}\pi\frac{D-p}{E}\sigma_p\gamma_p e^{r(T-t)} + \mu_p\right) \cdot \frac{D-2p}{E}(1 + \eta) + (1 - r)\frac{D-p}{E}(1 + \eta), \\
 B &= \mu_p \cdot \frac{D-p}{E} \cdot \frac{D-2p}{E} \cdot (1 + \eta),
\end{align*}

and

\begin{equation}
 C = p^2 \cdot \frac{D-p}{E} \cdot \sigma_p^2 \gamma_p e^{r(T-t)}(1 + \eta).
\end{equation}

Since for \(0 < a \leq 1\),

\begin{equation}
 D_1 = \lambda \frac{D-p}{E} \int_0^{+\infty} y e^{\gamma_p y} e^{r(T-t)} \gamma(y, \nu, \eta) dy = \lambda \frac{D-p}{E} \cdot \frac{1}{\left(a \gamma_p e^{r(T-t)} - 1\right)^2},
\end{equation}

and when we assume that the distribution of claim size is exponential with parameter \(\nu = 1\) and the density function of it is \(g(y) = e^{-y}\), then by putting equation (20) into (19), we get the implicit expression for \(a\) as follows:

\begin{equation}
 a(t) = \frac{C\left(\eta\left(\frac{D-2p}{E}\right)^2 - 1/E\right) + B - A - \lambda \frac{D-p}{E} \cdot \frac{1}{\left(a \gamma_p e^{r(T-t)} - 1\right)^2}}{(1 + \eta)C\left(\frac{D-2p}{E}\right)^2}.
\end{equation}

Since

\begin{align*}
 \frac{\partial A}{\partial p} &= (1 + \eta)\left(e^{r(T-t)}\right), \\
 \frac{\partial B}{\partial p} &= \mu_p (1 + \eta)\left(\frac{p(-3D + 4p)}{E^2} + \frac{D-p}{E} \cdot \frac{D-2p}{E}\right), \\
 \frac{\partial C}{\partial p} &= \sigma_p^2 \gamma_p e^{r(T-t)}(1 + \eta) \cdot \frac{2D-3p}{E},
\end{align*}
and
\[ \frac{\partial D_1}{\partial p} = \lambda \left( \frac{1}{E} - \gamma e^{r(T-t)} \right) \cdot \frac{D-p \partial \gamma}{E} \int_0^{+\infty} e^{\gamma y e^{r(T-t)}} G(dy), \]

where
\[ \int_0^{+\infty} e^{\gamma y e^{r(T-t)}} G(dy) = \frac{1}{(a \gamma e^{r(T-t)} - 1)^2} \]

then
\[ \frac{\partial a}{\partial p} = \frac{1}{(a \gamma e^{r(T-t)} - 1)} \]

Putting equations (17) through (22) into equation (16), we get an implicit function of \( p^*(t) \), but we cannot get the explicit solutions of \( (\pi^*(t), a^*(t), p^*(t)) \) by standard analytical methods. With Matlab and solving system equations of implicit functions (17) and (21), (18) and (22) by iterative method, we obtain the optimal solution of \( p^*(t) \) at any given values of time \( t \), \( 0 \leq t \leq T \). Substituting \( p^*(t) \) into (6), (18) and (19), we then obtain the optimal solutions of \( p^*(t), a^*, \) and \( \pi^* \), \( t = t_0, t_0 + \Delta t, ..., T \). When \( a^* > 1 \), we set \( a^* = 1 \). The function \( f(t) \) is determined by the following differential equation:

\[ f_t = \gamma e^{r(T-t)} \left( p^* \frac{D-p^*}{E} (a^*(1+\eta)-\eta) + (\mu-r)\pi^* - \left( rp^* \frac{D-p^*}{E} \mu_p p^* \frac{D-2p^*}{E} \right) (a(1+\eta)-\eta) \right) - \frac{1}{2} (0^2 + \sigma^2 \pi^2)(\gamma e^{r(T-t)})^2 - \left( \rho_{13} \pi \right) \frac{D-p^*}{E} \sigma \right) p^* (\gamma e^{r(T-t)})^2 \frac{D-2p^*}{E} (a^*(1+\eta) - \eta) + \mu_p p^* \frac{D-p^*}{E} \frac{D-2p^*}{E} \gamma e^{r(T-t)} (a^*(1+\eta) - \eta) - \frac{1}{2} p^* \frac{D-p^*}{E} \sigma_p^2 \gamma e^{r(T-t)} \left[ \left( \frac{D-2p^*}{E} \gamma e^{r(T-t)} \right)^2 + \frac{2}{E} \right] (a^*(1+\eta) - \eta)^2 + \lambda \frac{D-p^*}{E} \int_0^{+\infty} \left( e^{\gamma a^* y e^{r(T-t)}} - 1 \right) G(dy) \]
We show now that our solutions are global optimums based on the implicit function theorem. Putting equations (17), (18), (19), and (20) into equation (16) and writing it as \( H(t,p) = 0 \), we can find the Jacobian as:

\[
(DH)(c,d) = \left( \frac{\partial H}{\partial t}(c,d) \right) \left( \frac{\partial H}{\partial c}(c,d) \right) ,
\]

where \((c,d)\) is a point that satisfies \( G(c,d) = 0 \). Since \( G(t,p) \) is a combination of elementary functions, \( \frac{\partial H}{\partial t} \) and \( \frac{\partial H}{\partial p} \) are differentiable in the domain of function \( H(t,p) \). By the implicit function theorem, we find that we can write the form \( p = p(t) \) for all points where \( \frac{\partial H}{\partial p} \neq 0 \). The resulting solution of \( p^*(t) \) is sufficiently smooth so that a verification theorem can be used.

With the boundary condition of \( f(T) = 0 \), where \( \pi^*, a^*, \) and \( p^* \) satisfy the system of equations (16) through (22), from the above analysis, the following theorem is obtained:

**Theorem 2:** When the expected utility function of the terminal wealth of the insurer is exponential and the demand function for insurance products is linear, the optimal strategy \((\pi^*, a^*, p^*)\) is given by system of equations (16), (18), and (19), the optimal price \( q^* \) is determined as \( p^* = D - Eq^* \), and the optimal value function is:

\[
V(t,x,p) = -\exp(-\gamma(x + p^* q^*(a(1 + \eta) - \eta)))e^{-rt} + f(t) / \gamma ,
\]

where \( f(t) \) is given by equation (23).

From equation (17), we see that \( \pi^* \) is directly related to the drift rate \( \mu \) of the risky assets process, but inversely related to the volatility of return rate of risky assets \( \sigma \) and risk-free interest rate \( r \). Finally, \( \pi^* \) is inversely related to the coefficient of risk aversion \( \gamma \). More detailed numerical analysis of these relationships will be presented in the next section. It should be noted that the optimal solution \( \pi^* \) may be less than zero when \( D < 2p^* \) (see equation (17)), and in that case the optimal investment strategy is to short the risky assets, while investing more than 100\% of the portfolio in the risk-free asset.

The feasible solution interval for the retention rate of reinsurance is \([0, 1]\) and a retention rate of 1 means the insurer purchases no reinsurance (full retention). However, our optimal retention rate is smaller than 1, which means the optimal strategy for the insurer in our case is to purchase some reinsurance. That is, reinsurance can improve the insurer’s utility and performance.
NUMERICAL ANALYSIS

In our analysis we assume the following values of the parameters (based primarily on prior literature). The name and assumed values of the parameters are listed in Table 1.

Table 1. The Names and Assumed Values of Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>the drift of the return rate of risky assets invested by the insurer</td>
<td>0.10</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>the volatility of the return rate of risky assets invested by the insurer</td>
<td>0.10</td>
</tr>
<tr>
<td>$\theta$</td>
<td>the coefficient of jump diffusion of claim loss</td>
<td>0.15</td>
</tr>
<tr>
<td>$\rho_{13}$</td>
<td>the coefficient of correlation between insurance price and risky investment</td>
<td>0.20</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>the coefficient of risk aversion</td>
<td>2</td>
</tr>
<tr>
<td>$T$</td>
<td>the time horizon</td>
<td>20</td>
</tr>
<tr>
<td>$D$</td>
<td>the intercept of the linear function of insurance demand</td>
<td>0.5</td>
</tr>
<tr>
<td>$E$</td>
<td>the slope of the linear function of insurance demand</td>
<td>$1.0 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>the intensity of claim size</td>
<td>0.05</td>
</tr>
<tr>
<td>$\mu_p$</td>
<td>the drift of the average price of insurance market</td>
<td>0.05</td>
</tr>
<tr>
<td>$\sigma_p$</td>
<td>the volatility of the average price of insurance market</td>
<td>0.15</td>
</tr>
<tr>
<td>$r$</td>
<td>the risk-free interest rate</td>
<td>0.05</td>
</tr>
<tr>
<td>$\eta$</td>
<td>the rate of reinsurance cost</td>
<td>0.10</td>
</tr>
</tbody>
</table>

We obtain the following optimal solutions when time $t$ takes values of 1 through 20 listed in Table 2.

From Table 2 we see that optimal price and optimal amount of risky assets increases with time. However, the optimal quantity demanded and optimal proportion of retention decreases with time. An aggressive investment strategy corresponds with having less retention in order to hedge the investment risk. Next, we study the sensitivity of the model by varying the parameters.

Varying the Parameters $\mu$, $\sigma$, $\mu_p$, $\sigma_p$, and $r$

Figure 1 (a) through Figure 1 (o) display the change pattern in the values of the optimal price, $p^*$, the optimal risky assets, $\pi^*$, and the optimal proportion of retention, $a$, with changes in the return rate of the risky assets
Table 2. Optimal solutions for varying time $t$

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^*$</td>
<td>0.1578</td>
<td>0.1607</td>
<td>0.1637</td>
<td>0.1666</td>
<td>0.1694</td>
<td>0.1722</td>
<td>0.1749</td>
<td>0.1776</td>
</tr>
<tr>
<td>$\pi^*$</td>
<td>48.4231</td>
<td>50.7658</td>
<td>53.2217</td>
<td>55.7964</td>
<td>58.4959</td>
<td>61.3266</td>
<td>64.2956</td>
<td>67.4101</td>
</tr>
<tr>
<td>$a^*(10^{-1})$</td>
<td>0.9064</td>
<td>0.9063</td>
<td>0.9060</td>
<td>0.9058</td>
<td>0.9055</td>
<td>0.9053</td>
<td>0.9050</td>
<td>0.9047</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$t$</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^*$</td>
<td>0.1802</td>
<td>0.1827</td>
<td>0.1852</td>
<td>0.1877</td>
<td>0.1900</td>
<td>0.1924</td>
<td>0.1946</td>
<td>0.1968</td>
</tr>
<tr>
<td>$\pi^*$</td>
<td>70.6782</td>
<td>74.1084</td>
<td>77.7100</td>
<td>81.4930</td>
<td>85.4679</td>
<td>89.6467</td>
<td>94.0419</td>
<td>98.6675</td>
</tr>
<tr>
<td>$a^*(10^{-1})$</td>
<td>0.9043</td>
<td>0.9040</td>
<td>0.9036</td>
<td>0.9031</td>
<td>0.9027</td>
<td>0.9021</td>
<td>0.9016</td>
<td>0.9010</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$t$</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^*$</td>
<td>0.1989</td>
<td>0.2010</td>
<td>0.2030</td>
<td>0.2050</td>
</tr>
<tr>
<td>$q^*(10^4)$</td>
<td>3.0107</td>
<td>2.9899</td>
<td>2.9698</td>
<td>2.9502</td>
</tr>
<tr>
<td>$\pi^*$</td>
<td>103.5388</td>
<td>108.6727</td>
<td>114.0879</td>
<td>119.8056</td>
</tr>
<tr>
<td>$a^*(10^{-1})$</td>
<td>0.9003</td>
<td>0.8996</td>
<td>0.8988</td>
<td>0.8979</td>
</tr>
</tbody>
</table>

Note: $p^*$: optimal price of insurance products; $q^*$: optimal quantity demanded of insurance products; $\pi^*$: optimal portion of risky investment; $a^*$: optimal portion of retention of reinsurance.

invested, $\mu$, the standard deviation of the risky assets invested, $\sigma$, the drift and volatility of insurance price, $\mu_p$ and $\sigma_p$, and the risk-free interest rate, $r$. The curve surfaces in Figures 1 (a), (b), and (c) show that the optimal price does not change regardless of the value of $\mu$. Increasing the value of $\mu$ will slightly increase the optimal amount of risky assets, while the optimal proportion of retention will not be affected by changing the value of $\mu$.

Figures 1 (d) and (f) show that the optimal value of $p^*$ and the optimal portion of retention $a^*$ does not change regardless of the value of the volatility of risky assets, $\sigma$. Figure 1 (e) shows that the optimal value of $\pi^*$ decreases with the increase of volatility of risky assets, $\sigma$, and the optimal value of $\pi^*$ is rather sensitive to the change of $\sigma$. Considering equation (17), we see that where the value of the second term is much larger than the first term and $\mu$ only appears in the first term but $\sigma$ affects both terms,
it is clear that the effect of changes in $\mu$ on optimal investment strategy is rather small, especially when the optimal quantity demanded is large. This is in contrast to Lin and Li (2011), where they assume the quantity demanded is one unit and the premium rate is a constant. Therefore, in their situation, both $\mu$ and $\sigma$ affect the optimal investment strategy greatly.

Generally speaking, insurance prices and also likely reinsurance prices/practices are affected by the riskiness of insurer (Sommer, 1996). However, insurance price also will be affected by quantity demanded, and quantity demanded can also play a similar role as reinsurance to reduce underwriting risk but without reinsurance cost. Therefore, whether optimal solutions of price, amount of risky assets invested, and the proportion of reinsurance are affected by parameters of $\mu$ and $\sigma$ is dependent on the balance among all important factors.

From Figures 1 (g) through (i) we see that increasing the insurance price $\mu_p$ or decreasing the volatility of insurance price $\sigma_p$, will increase optimal price and amount of risky assets and decrease optimal portion of retention, which means lower price risk will promote the insurer to take more aggressive investment strategy. And the optimal solutions of $p^*$ and $\pi^*$ are very sensitive to the change of the drift and volatility of insurance price, meaning that pricing risk (the fluctuation of the drift and volatility of insurance price) can be hedged by selecting optimal strategies of pricing and investment. But the optimal retention $a^*$ is not sensitive to the change of parameters of $\mu_p$ and $\sigma_p$.

From Figure 1 (m) and (n) we find that the optimal price and the optimal amount of risky assets invested decrease as the risk-free interest rate increases, and optimal solutions of $p^*$ and $\pi^*$ are rather sensitive to the change of $r$. From Figure 1 (o), we find that the optimal retention $a^*$ slightly increases with increases in $r$. The safety of the higher risk-free interest rate will encourage the insurer to increase the amount of retention.

**Varying the Parameters $\eta$, $\gamma$, $\lambda$, and $\rho_{13}$**

Next, we also study the sensitivity of the model by varying the parameters of the rate of reinsurance, $\eta$, the coefficient of risk aversion, $\gamma$, the intensity of claim size, $\lambda$, and the correlation coefficient between insurance price and risky assets invested, $\rho_{13}$, as illustrated in Figure 2 (a) through Figure 2 (l). The plots show that the optimal price and the optimal risky assets invested in Figures 2 (a) and (b) remain unchanged even though the ratio of the reinsurance cost changes. From Figure 2 (c) we find that optimal proportion of retention increases with increases in the cost of reinsurance. Figures 2 (d) and 2 (e) show that the coefficient of risk aversion, $\gamma$, will affect the optimal solutions of $p^*$, $\pi^*$, and $a^*$. Increasing the value of $\gamma$
Fig. 1 (a). Change in the optimal price of insurance products $p^\ast$ as the means of return rate of risky assets $\mu$ and time $t$ change.

Fig. 1 (b). Change of the optimal risky assets of insurer $\pi^\ast$ as the means of return rate of risky assets $\mu$ and time $t$ change.
**JUMP-DIFFUSION RISK PROCESS**

\[ r = 0.05, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15, \]
\[ \gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda, = 1, \eta = 0.1, \theta = 0.15 \]

**Fig. 1 (c).** Optimal proportion of retention \( a^* \) as the means of return rate of risky assets \( \mu \) and time \( t \) change.

\[ r = 0.05, \mu = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15, \]
\[ \gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda, = 1, \eta = 0.1, \theta = 0.15 \]

**Fig. 1 (d).** Optimal price of insurance products \( p^* \) as the volatility of risky assets \( \sigma \) and time \( t \) change.
Fig. 1 (e). Optimal amount of risky assets $\pi^*$ as the volatility of risky assets $\sigma$ and time $t$ change.

$$r = 0.05, \mu = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15,$$
$$\gamma = 2, \ T = 20, \ D = 0.5, \ E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1, \theta = 0.15$$

Fig. 1 (f). Optimal proportion of retention $\alpha^*$ as the volatility of risky assets $\sigma$ and time $t$ change.

$$r = 0.05, \mu = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15,$$
$$\gamma = 2, \ T = 20, \ D = 0.5, \ E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1, \theta = 0.15$$
Fig. 1 (g). Optimal price $p^*$ as the drift of insurance price $\mu_p$ and time $t$ change.

$$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \sigma_p = 0.15,\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1, \theta = 0.15$$

Fig. 1 (h). Optimal amount of risk assets $\pi^*$ as the drift of insurance price $\mu_p$ and time $t$ change.

$$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \sigma_p = 0.15,\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1, \theta = 0.15$$
Fig. 1 (i). Optimal proportion of retention $a^*$ as the drift of insurance price $\mu_p$ and time $t$ change.

$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \sigma_p = 0.15, \gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1, \theta = 0.15$

Fig. 1 (j). Optimal price $p^*$ as the volatility of insurance price $\sigma_p$ and time $t$ change.

$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1, \theta = 0.15$
Fig. 1 (k). Optimal amount of risk assets $\pi^*$ as the volatility of insurance price $\sigma_p$ and time $t$ change.

$$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05,$$
$$\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1, \theta = 0.15$$

Fig. 1 (l). Optimal proportion of retention $a^*$ as the volatility of insurance price $\sigma_p$ and time $t$ change.

$$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05,$$
$$\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1, \theta = 0.15$$
Fig. 1 (m). Optimal price $p^*$ as the risk-free interest rate $r$ and time $t$ change.

Fig. 1 (n). Optimal amount of risky assets $\pi^*$ as the risk-free interest rate $r$ and time $t$ change.
will decrease the optimal price and decrease the proportion of risky assets. And the decrease of the optimal price will increase the quantity demanded so as to hedge the underwriting risk. From Figure 2 (f) we find that optimal proportion of retention slightly increases with the increase of $\gamma$. Because higher quantity demanded hedges the underwriting risk, the insurer can increase the proportion of retention.

From Figures 2 (g) and (h), we find that optimal price and optimal amount of risky assets invested are directly related to the parameter $\lambda$, which means that higher claim size leads to higher optimal insurance price. With the insurer collecting more funds from higher premium income, this will encourage the insurer to select a more aggressive investment strategy. Figure 2 (i) illustrates that increasing $\lambda$ will slightly decrease the optimal proportion of retention in order to decrease the underwriting risk. However, from equation (21) we know there are several factors affecting the proportion of retention. Higher quantity demanded (the optimal quantity demanded, $\frac{D-p}{E} = 2.9482 \times 10^4$ when $t = 20$ in our case) will automatically hedge underwriting risk so $\lambda$ has very small effect on the retention proportion. However, it does not imply that other factors will not affect the optimal proportion of retention.
The results are distinct from those in Lin and Li (2011) where the premium rate is assumed to be a constant. In their paper, the optimal retention decreases greatly when the claim intensity $\lambda$ increases. Since the premium rate is a constant, all claim risk due to the increase of $\lambda$ is transferred by reinsurance. Therefore, the optimal proportion of retention is very sensitive to the change of $\lambda$ in their case. We believe that it is more realistic to allow the premium rate to vary, since the price of insurance is affected by several factors, including quantity demanded of insurance market, underwriting risk, investment risk, time, and other factors.

From Figures 2 (j), (k) and (l), we find that the correlation coefficient between insurance price and risky assets invested, $\rho_{13}$, has little influence on the optimal price and optimal retention proportion, but has an important effect on the optimal investment strategy. Greater positive value of $\rho_{13}$ will result in higher optimal amounts invested in risky assets. Figure 2 (m) shows that negative value of $\rho_{13}$ will result in short position of risky assets. Table 3 summarizes the change directions of optimal solutions when the parameters of $\mu$, $\sigma$, $\mu_p$, $\sigma_p$, $r$, $\eta$, $\gamma$, $\lambda$, and $\rho_{13}$ change.

From Table 3, we observe that the retention level is not very sensitive to most parameters except for the time parameter and the gross reinsurance premium. However, we see the optimal price is sensitive to most parameters except the drift and volatility of the rate of return on the risky investment and gross reinsurance premium. The optimal amount of risky investment is sensitive to most parameters except the drift of rate of return on the risky investment and gross reinsurance premium. The findings illustrate that an insurer’s pricing strategy can decrease pricing risk and underwriting risk. Meanwhile, a sound investment strategy can decrease the risk of return volatility on risky investments and underwriting risk. The optimal reinsurance strategy mainly depends on time and the cost of reinsurance.

**Empirical Implications of the Findings**

Based on the above analysis, three empirical implications stand out as particularly relevant, especially in the low interest rate environment following the Great Recession. The analysis suggests that as the risk-free rate decreases, we should observe increases in (a) insurance prices, (b) the amount of risky assets held by insurers, and (c) the proportion of risk that is reinsured. Anecdotal evidence in support of the first two predictions is discussed in a survey by Towers Watson (2012), and the third prediction appears to be a fruitful area for further research.
Fig. 2 (a). Optimal price of insurance products $p^*$ as the ratio of reinsurance cost $\eta$ and time $t$ change.

**Figure 2 (b).** Optimal amount of risk assets $\pi^*$ as the ratio of reinsurance cost $\eta$ and time $t$ change.
\[ \begin{align*}
\sigma_D &= 0.10, \mu = 0.10, \rho_{12} = -0.1, \rho_{13} = -0.2, \rho_{23} = 0.2, \mu_p = 0.05, \rho_1 = 0.1, \\
\sigma_p &= 0.10, \sigma = 0.10, \gamma = 2, T = 8, D = 0.22, E = 1.0 \times 10^{-5}, \theta = 0.15
\end{align*} \]

**Fig. 2 (c).** Optimal proportion of retention \( a^* \) as the risk-free interest rate \( r \) and time \( t \) change.

\[ \begin{align*}
r &= 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15, \\
T &= 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1, \theta = 0.15
\end{align*} \]

**Fig. 2 (d).** Optimal price of insurance products \( p^* \) as the coefficient of risk aversion \( \gamma \) and time \( t \) change.
Fig. 2 (e). Optimal amount of risk assets $\pi^*$ as the coefficient of risk aversion $\gamma$ and time $t$ change.

Fig. 2 (f). Optimal proportion of retention $a^*$ as the coefficient of risk aversion $\gamma$ and time $t$ change.
Fig. 2 (g). Optimal price of insurance products $p^*$ as the intensity of claim $\lambda$ and time $t$ change.

$$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15,$$
$$\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \eta = 0.1, \theta = 0.15$$

Fig. 2 (h). Optimal amount of risky assets $\pi^*$ as the intensity of claim $\lambda$ and time $t$ change.

$$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15,$$
$$\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \eta = 0.1, \theta = 0.15$$
\[ r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15, \]
\[ \gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \eta = 0.1, \theta = 0.15 \]

**Fig. 2 (i).** Optimal proportion of retention \( p^* \) as the the intensity of claim \( \lambda \) and time \( t \) change.

\[ r = 0.05, \mu = 0.10, \sigma = 0.10, \mu_p = 0.05, \sigma_p = 0.15, \]
\[ \gamma = 2, T = 20, \lambda = 1, D = 0.5, E = 1.0 \times 10^{-5}, \eta = 0.1, \theta = 0.15 \]

**Fig. 2 (j).** Optimal price of insurance products \( p^* \) as the correlation coefficient of \( \rho_{13} \) and time \( t \) change.
Fig. 2 (k). Optimal amount of risky assets \( \pi^* \) as the correlation coefficient of \( \rho_{13} \) and time \( t \) change (\( \rho_{13} \geq 0 \)).

\[
\begin{align*}
    r &= 0.05, \mu = 0.10, \sigma = 0.10, \mu_p = 0.05, \sigma_p = 0.15, \\
    \gamma &= 2, T = 20, \lambda = 1, D = 0.5, E = 1.0 \times 10^{-5}, \eta = 0.1, \theta = 0.15
\end{align*}
\]

Fig. 2 (l). Optimal proportion of retention \( a^* \) as the correlation coefficient of \( \rho_{13} \) and time \( t \) change (\( \rho_{13} \geq 0 \)).

\[
\begin{align*}
    r &= 0.05, \mu = 0.10, \sigma = 0.10, \mu_p = 0.05, \sigma_p = 0.15, \\
    \gamma &= 2, T = 20, \lambda = 1, D = 0.5, E = 1.0 \times 10^{-5}, \eta = 0.1, \theta = 0.15
\end{align*}
\]
SUMMARY AND CONCLUSIONS

Based on stochastic optimal control theory and a stochastic dynamic optimization model, we find the optimal price, reinsurance, and investment strategy for an insurer. We establish a pricing model with the assumptions that the price, the investment, and the claim loss rate are stochastic processes, and we also assume that the insurance price is a linear function of demand in a competitive insurance market. The objective of our model is to maximize the expected utility of the insurer’s terminal wealth. We also study the sensitivity of the models by varying the parameters. The results show that the shift and volatility of insurance price, the parameter of the intensity of claim size and other parameters will affect the insurer’s optimal price, reinsurance, and investment strategy.

This work has possible applications for insurers seeking an optimal dynamic management strategy in the context of using a model similar to the one examined here as an enterprise risk management tool. Regulatory developments, including the Own Risk and Solvency Assessment in the United States, and Solvency II in the European Union, encourage or even require such an integrated risk management approach. Our model may be further extended by considering more complex structures of risks such as

Fig. 2 (m). Optimal amount of risky assets \( \pi^* \) as the correlation coefficient of \( \rho_{13} \) and time \( t \) change \( (\rho_{13} \leq 0) \).
multiple risky asset classes and relaxing our restrictive assumptions about the market price process, utility of wealth, and demand for insurance. The model may also be further extended by considering the correlation between price and claim intensity to obtain optimal solutions by simulation and numerical methods.

**REFERENCES**


